

Approximations of Lipschitz maps via immersions and differentiable exotic sphere theorems^{*†}

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April 12, 2016

Abstract

As our main theorem, we prove that a Lipschitz map from a compact Riemannian manifold M into a Riemannian manifold N admits a smooth approximation via immersions if the map has no singular points on M in the sense of F.H. Clarke, where $\dim M \leq \dim N$. As its corollary, we have that if a bi-Lipschitz map between compact manifolds and its inverse map have no singular points in the same sense, respectively, then they are diffeomorphic. We have three applications of the main theorem: The first two ones are two differentiable sphere theorems for a pair of topological spheres including that of exotic ones. The third one is that a compact n -manifold M is a twisted sphere and there exists a bi-Lipschitz homeomorphism between M and the unit n -sphere $S^n(1)$ which is a diffeomorphism except for a single point, if M satisfies certain two conditions with respect to critical points of its distance function in the Clarke sense. Moreover, we have three corollaries from the third theorem; the first one is that for any twisted sphere Σ^n of general dimension n , there exists a bi-Lipschitz homeomorphism between Σ^n and $S^n(1)$ which is a diffeomorphism except for a single point. In particular, there exists such a map between an exotic n -sphere Σ^n of dimension $n > 4$ and $S^n(1)$; the second one is that if an exotic 4-sphere Σ^4 exists, then Σ^4 does not satisfy one of the two conditions above; the third one is that for any Grove-Shiohama type n -sphere N , there exists a bi-Lipschitz homeomorphism between N and $S^n(1)$ which is a diffeomorphism except for one of points that attain their diameters.

1 Introduction

1.1 Motivations and our main theorem

We will first mention our motivations behind our purpose, which are exotic structures: No one needs the introduction of exotic n -dimensional spheres Σ^n , however Σ^n were very first

^{*}2010 Mathematics Subject Classification: Primary 49J52, 53C20; Secondary 57R12, 57R55.

[†]Key words and phrases: bi-Lipschitz homeomorphism, differentiable sphere theorem, exotic spheres, Lipschitz map, non-smooth analysis, smooth approximation.

[‡]Partly supported by the two Grant-in-Aids for Science Reserch (C), JSPS KAKENHI Grant Numbers 15K04846 and 16K05133.

discovered by Milnor [25] in the case where $n = 7$, which is, by definition, homeomorphic to the standard n -sphere S^n but not diffeomorphic to it. Note that the smooth 4-dimensional Poincaré conjecture (SPC4), that is, the problem of the existence of an exotic structure on the 4-sphere, is still open.

Due to exotic structures, we always have technical difficulties when investigating whether a topological sphere theorem can be a differentiable sphere theorem as its generalization. The difficulties become clearer from global Riemannian geometry's stand point: It follows from Smale's h-cobordism theorem [35] together with [34] that every homotopy sphere of dimension $n \geq 5$ is a twisted sphere, which is a smooth manifold obtained by glueing two standard n -discs along their boundaries under a boundary diffeomorphism. This implies that every Σ^n ($n > 4$) is actually twisted, since Σ^n is a homotopy sphere. Applying Weinstein's deformation technique for metrics ([37, Proposition C]) to two discs embedded smoothly into a twisted sphere X of general dimension, respectively, we see that X admits a metric such that the cut locus¹ of some point on X is a single point. By all together above, we have

Theorem 1.1 (Also see [2, Proposition 7.19]) *Every exotic sphere Σ^n of dimension $n > 4$ admits a metric such that there is a point whose cut locus is one point.*

Thus, it is very hard for us to notice the difference between Σ^n and S^n from the point of view of two exponential maps at the single point on Σ^n and any point on S^n , respectively. For example, one of open problems in global Riemannian geometry is if the Grove-Shiohama type n -sphere can be diffeomorphic to S^n . Here, the Grove-Shiohama type sphere means a complete Riemannian manifold V satisfying sectional curvature $K_V \geq 1$ and its diameter $\text{diam}(V) > \pi/2$ ([18]). Since such a V is twisted, from Theorem 1.1 we can infer that some single cut point on V is a big obstacle as a singular one whenever approximating a homeomorphism, in fact which is bi-Lipschitz, between V and S^n by diffeomorphisms at the cut point.

Hence, it is a cardinal importance to do an analysis of such singular points on an arbitrary manifold. For this, we employ a notion used in non-smooth analysis of F.H. Clarke ([5], [6]), i.e., a non-singular point for a Lipschitz function/map in this article. The following example shows that the analysis is a strong tool in differentiable geometry:

Example 1.2 Let M be a complete Riemannian manifold, d the distance function of M . Take any point $p \in M$, and fix it. Set $d_p(x) := d(p, x)$ for all $x \in M$. Then, the point $q \in M \setminus \{p\}$ is a *critical point of d_p* (or *critical point for p*) in the sense of Grove-Shiohama [18], if for every nonzero tangent vector $v \in T_q M$ at q , there exists a minimal geodesic segment γ emanating from q to p such that

$$\angle(v, \dot{\gamma}(0)) \leq \frac{\pi}{2}.$$

¹The cut locus $\text{Cut}(p)$ of a point p in a complete Riemannian manifold M is, by definition, the closure of the set of all points $x \in M$ such that there are at least two minimal geodesics emanating from p to x . Then, a point in $\text{Cut}(p)$ is called the cut point of p . For example, $\text{Cut}(N)$ of $N := (0, \dots, 0, 1) \in S^n(1)$ is $\{(0, \dots, 0, -1)\}$, where $S^n(1) := \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\}$. Note that the distance function to a point x of M is not differentiable at the cut point of x .

Here, $\angle(v, \dot{\gamma}(0))$ denotes the angle between v and $\dot{\gamma}(0)$. Note that a critical point of d_p is the cut point of p . Assume that, for some $r > 0$, $\partial B_r(p) := \{x \in M \mid d(p, x) = r\}$ has no critical points of d_p . By Gromov's isotopy lemma [15], $\partial B_r(p)$ is a topological submanifold of M . Since $\partial B_r(p)$ is also free of critical points of d_p in the sense of Clarke (See Example 1.9), it follows from Clarke's implicit function theorem [7] that

in fact, $\partial B_r(p)$ is a *Lipschitz* submanifold of M .

We are going to give the definition of the Clarke sense below (See Definition 1.5 in Sect 1.2).

Our purpose of this article is *to establish an approximation method for a Lipschitz map via diffeomorphisms using the notion of non-smooth analysis and to apply this method to prove differentiable sphere theorems*. That is, our main theorem is as follows:

Theorem 1.3 (Main Theorem) *Let $F : M \longrightarrow N$ be a Lipschitz map from a compact Riemannian manifold M into a Riemannian manifold N , where $\dim M \leq \dim N$. If F has no singular points on M (in the sense of Clarke), then for any $\eta > 0$, there exists a smooth immersion f_η from M into N such that*

$$\max_{x \in M} d_N(F(x), f_\eta(x)) < \eta, \quad \text{Lip}(f_\eta) \leq \text{Lip}(F)(1 + \eta).$$

Here, $\text{Lip}(f_\eta)$ and $\text{Lip}(F)$ denote the Lipschitz constants of f_η and F , respectively, i.e.,

$$\text{Lip}(f_\eta) := \sup \left\{ \frac{d_N(f_\eta(x), f_\eta(y))}{d_M(x, y)} \mid x, y \in M, x \neq y \right\},$$

where d_M and d_N are the distance functions of M and N , respectively.

Remark 1.4 We give three remarks on Theorem 1.3.

- The definition of a *singular point* for a Lipschitz map, i.e., non-smooth analysis of Clarke, is given in Sect 1.2.
- It should be emphasized that the reason why Clarke introduced the notion of non-smooth analysis was not for approximations of a Lipschitz map via diffeomorphisms, but for the inverse function theorem for a Lipschitz map, which contains the classical one as a special case. See also Example 1.8 below.
- Our approximation method for Lipschitz maps to prove Theorem 1.3 generalizes the whole Grove-Shiohama one in [18]. See Sect. 2.

As an indirect corollary of Theorem 1.3, for any twisted sphere Σ^n of any dimension n , we can construct a concrete bi-Lipschitz homeomorphism F between Σ^n and S^n which is a diffeomorphism except for a single point (See Corollary 1.15). Therefore, by this result together with Theorem 1.1, the existence of exotic n -spheres ($n > 4$) implies that we cannot approximate F by diffeomorphisms. Hence, we must give careful consideration to sufficient conditions for a pair of topological spheres admitting a single cut point of some point, hence even for that of exotic ones, to be diffeomorphic.

As applications of Theorem 1.3, we prove differentiable sphere theorems not only for such a pair above, but for a pair of Σ^n and S^n , where we give the sufficient conditions for each pair to be diffeomorphic. See Sect. 1.3.

1.2 Non-smooth analysis and a corollary of Theorem 1.3

F.H. Clarke very first established the non-smooth analysis in [5] and [6]. It is a strong tool not only in the optimal control theory (cf. [8]), but also in differentiable geometry (e.g., [20], [30], and Example 1.2): Let M, N be Riemannian manifolds, respectively, and let $F : M \rightarrow N$ be a Lipschitz map. In the case where $N = \mathbb{R}$, let $f := F : M \rightarrow \mathbb{R}$. By Rademacher's theorem [29], there exists a set $E_F \subset M$ of measure zero such that the differential dF of F exists on $M \setminus E_F$. Then, for each point x , there exists a sequence $\{x_i\}$ of $x_i \in M \setminus E_F$ convergent to x , and hence we can define the *generalized differential* $\partial F(x)$ of F at $x \in M$ as follows:

$$(1.1) \quad \partial F(x) := \text{Conv}(\{\lim_{i \rightarrow \infty} dF_{x_i} \mid dF_{x_i} \text{ exists as } x_i \in M \setminus E_F \rightarrow x\}),$$

where “Conv(\cdot)” means “convex hull”. In case $N = \mathbb{R}$, we call $\partial f(x)$ the *generalized gradient of f at x* , i.e.,

$$\partial f(x) := \text{Conv}(\{\lim_{i \rightarrow \infty} \nabla f(x_i) \mid \nabla f(x_i) \text{ exists as } x_i \in M \setminus E_f \rightarrow x\}).$$

Here ∇f denotes the gradient vector field of f . Note that both definitions above do not depend on atlases (e.g., [20]), and that $\partial F(x)$, $\partial f(x)$ are compact convex sets.

Definition 1.5 ([5], [6]) Let M, N be the same above, and $U \subset M$ an open set.

- A point $x \in U$ is called *non-singular of a Lipschitz map $F : U \rightarrow N$* , if every element in $\partial F(x)$ is of maximal rank.
- A point $x \in U$ is called *non-critical of a Lipschitz function $f : U \rightarrow \mathbb{R}$* , if

$$0 \notin \partial f(x),$$

where 0 denotes the zero tangent vector at x .

We will give several examples of Definition 1.5 with a few related remarks.

Example 1.6 Consider two functions $f_1(x) := x^2$, $f_2(x) := x + 2$ on $(-2, 3)$, respectively. Define a Lipschitz function $f(x) := \max\{f_1(x), f_2(x)\}$ on $(-2, 3)$, i.e.,

$$f(x) = \begin{cases} x^2 & \text{on } (-2, -1) \cup (2, 3), \\ x + 2 & \text{on } [-1, 2]. \end{cases}$$

Note that f is not differentiable at $x = -1, 2$. Since

$$\lambda \frac{df_1}{dx}(-1) + (1 - \lambda) \frac{df_2}{dx}(-1) = -3\lambda + 1$$

for all $\lambda \in [0, 1]$, we have $\partial f(-1) = [-2, 1]$. As well as above, we have $\partial f(2) = [1, 4]$. Since $0 \in \partial f(-1)$ and $0 \notin \partial f(2)$, $x = -1$ is a critical point of f and $x = 2$ is not that of f . Note that $f(-1) = 1$ is the minimum value of f .

Example 1.7 ([6, Remark 1]) Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a Lipschitz map defined by $F(x, y) := (|x| + y, 2x + |y|)$. Note that F is not differentiable at $(x, y) = (0, 0)$. Then,

$$\begin{aligned} \partial F(0, 0) &= \text{Conv} \left(\left\{ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \right\} \right) \\ &= \left\{ \begin{pmatrix} s & 1 \\ 2 & t \end{pmatrix} \mid |s| \leq 1, |t| \leq 1 \right\}. \end{aligned}$$

Thus, $(0, 0)$ is non-singular of F .

Example 1.8 ([6, Lemmas 3, 4]) Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a map that is Lipschitz near some point $p \in \mathbb{R}^n$, where $n \leq k$. Assume that p is non-singular of F . Then for any $u \in S^{n-1}(1)$, $\partial F(p)u := \{Au \mid A \in \partial F(p)\}$ is convex in \mathbb{R}^k and $0 \notin \partial F(p)u$. Hence, there exist a vector $v \in S^{k-1}(1)$ and a number $\delta(p) > 0$ such that

$$\langle Au, v \rangle \geq 2\delta(p)$$

for all $A \in \partial F(p)$. Moreover, by the definition of $\partial F(p)$, we may find a number $r(p) > 0$ satisfying

$$\langle Au, v \rangle \geq \delta(p)$$

for all $A \in \partial F(q)$ and all $q \in B_{r(p)}(p) := \{x \in \mathbb{R}^n \mid \|p - x\| < r(p)\}$. Furthermore, one can prove that if $q_1, q_2 \in \overline{B_{r(p)}(p)} := \{x \in \mathbb{R}^n \mid \|p - x\| \leq r(p)\}$, F is increasing near p , that means that

$$\|F(q_2) - F(q_1)\| \geq \delta(p)\|q_2 - q_1\|.$$

Using this inequality, Clarke proved the existence of a local Lipschitzian inverse at a non-singular point of a Lipschitz map (See [6, Theorem 1]).

Example 1.9 (Critical points of Grove-Shiohama [18]) Let M be a complete Riemannian manifold, d the distance function of M , and fix $p \in M$. Assume that $q \in M$ is not a critical point for p in the sense of Grove-Shiohama (Refer to Example 1.2 for its definition). By definition, there exists $w \in T_q M \setminus \{0\}$ such that $\angle(w, \dot{\gamma}(0)) > \pi/2$ holds for all minimal geodesic segments γ emanating from q to p . Hence, $0 \notin \partial d_p(q)$, where $d_p(x) := d(p, x)$ for all $x \in M$. Note that a point $q \neq p$ is a critical point of d_p if and only if $0 \in \partial d_p(q)$. Since $\partial d_p(p)$ equals the unit closed ball centered at the origin of $T_p M$, $0 \in \partial d_p(p)$. Note that, even if $0 \notin \partial d_p(q)$, it is possible to occur that q is a cut point of p in general.

As a direct consequence of Theorem 1.3, we have

Corollary 1.10 *Let F be a bi-Lipschitz homeomorphism from a compact Riemannian manifold M onto a Riemannian manifold N . If F and F^{-1} have no singular points on M and N , respectively, then M and N are diffeomorphic.*

Remark 1.11 Shikata would be the first researcher who approximated a bi-Lipschitz homeomorphism via diffeomorphisms: In [31], he introduced a distance for a pair of compact differentiable manifolds which are bi-Lipschitz homeomorphic and proved that if the distance between such a pair is smaller than a certain positive constant, then

the bi-Lipschitz map can be approximated via diffeomorphisms, i.e., the manifolds are diffeomorphic. Moreover, as an application to the differentiable pinching problem, he proved in [32] that there exists a certain constant $\delta(n) \in (1/4, 1)$ depending on a number n such that if sectional curvature of a simply connected, compact Riemannian manifold M of dimension n is $\delta(n)$ -pinched, then M is diffeomorphic to the standard sphere. What astonishes us is that he defined such a distance between two manifolds for getting the differentiable sphere theorem more than ten years before Gromov's Hausdorff distance in [16].

1.3 Applications of Theorem 1.3: Three differentiable sphere theorems

Now, we are going to state applications of Theorem 1.3: Taking Theorem 1.1 into account, we consider the following setting. Let M_i ($i = 1, 2$) be a compact manifold of dimension n admitting a point $p_i \in M_i$ with a single cut point $q_i \in M_i$, and let $d_{M_i}(p_i, q_i) = \ell_i$, where d_{M_i} denotes the distance function of M_i . Note that M_i is homeomorphic to S^n . Choose a linear isometry $I : T_{p_1}M_1 \rightarrow T_{p_2}M_2$, where $T_{p_i}M_i$ denotes the tangent space of M_i at p_i . For each $i = 1, 2$, let $\sigma_{q_i}^{p_i}$ be a diffeomorphism from $\mathbb{S}_{p_i}^{n-1} := \{v \in T_{p_i}M_i \mid \|v\| = 1\}$ onto $\mathbb{S}_{q_i}^{n-1}$ defined by

$$(1.2) \quad \sigma_{q_i}^{p_i}(u_i) := -\dot{\tau}_{u_i}(\ell_i),$$

where $\tau_{u_i}(t) := \exp_{p_i} t u_i$ for all $u_i \in \mathbb{S}_{p_i}^{n-1}$ and all $t \in [0, \ell_i]$. Thus, we have a diffeomorphism

$$(1.3) \quad \sigma : \mathbb{S}_{q_1}^{n-1} \rightarrow \mathbb{S}_{q_2}^{n-1}$$

defined by $\sigma := \sigma_{q_2}^{p_2} \circ I \circ \sigma_{p_1}^{q_1}$, where $\sigma_{p_1}^{q_1} := (\sigma_{q_1}^{p_1})^{-1}$. The bi-Lipschitz constant $\text{Lip}^b(\sigma)$ of σ is given by

$$(1.4) \quad \text{Lip}^b(\sigma) := \inf\{\ell \mid \ell^{-1}\|u - v\| \leq \|\sigma(u) - \sigma(v)\| \leq \ell\|u - v\| \text{ for all } u, v \in \mathbb{S}_{q_1}^{n-1}\}.$$

For a geodesic segment $\gamma : [0, \pi] \rightarrow \mathbb{S}_{q_1}^{n-1}$ with $\|\dot{\gamma}\| := \|d\gamma/dt\| \equiv 1$, let $c : [0, \pi] \rightarrow \mathbb{S}_{q_2}^{n-1}$ be a curve defined by

$$(1.5) \quad c := \sigma \circ \gamma.$$

Then, we find that the differentiable structure between M_1 and M_2 depends on the map σ and the curve c above, i.e., the first application of Theorem 1.3 is stated as follows:

Differentiable Exotic Sphere Theorem I. *If $\text{Lip}^b(\sigma)$ and $\|\ddot{c}\| := \|d^2c/dt^2\|$ satisfy*

$$(1.6) \quad \text{Lip}^b(\sigma)^{-2} \geq 1 - \left\{ \frac{\sqrt{2} - 1}{2(e^\pi - 1)} \right\}^2 \quad \text{and} \quad \|\ddot{c}\|^2 \leq \text{Lip}^b(\sigma)^{-2} + \left\{ \frac{\sqrt{2} - 1}{2(e^\pi - 1)} \right\}^2$$

for all geodesic segments $\gamma([0, \pi]) \subset \mathbb{S}_{q_1}^{n-1}$ with $\|\dot{\gamma}\| \equiv 1$, or if the σ and n satisfy

$$(1.7) \quad \text{Lip}^b(\sigma)^2 \leq 1 + \left\{ \frac{8}{\pi}(n - 1) \right\}^{-\frac{1}{2}},$$

then M_1 and M_2 are diffeomorphic.

Remark 1.12 We give two remarks on the differentiable exotic sphere theorem I and the outlines of the proof of it in order to show Theorem 1.3 to be useful:

- The assumption (1.6) does not depend on the dimension. The first inequality in (1.6) implies that σ is almost an isometry, and makes the second one in (1.6) well-defined. (1.6) implies that c is almost a geodesic segment on $\mathbb{S}_{q_2}^{n-1}$. The inevitability of the restriction of \tilde{c} will be discussed in Sect. 3.4.
- The right side of (1.7) is the same constant as Karcher [21] estimated to get a sharper version of Shikata's theorem in [31].
- The outline of the proof in the case where (1.6): Let $d_{M_i}(p_i, q_i) := \pi$ ($i = 1, 2$) by normalizing the metric. For each $(t, u_1) \in [0, \pi] \times \mathbb{S}_{p_1}^{n-1}$, we define a bi-Lipschitz homeomorphism $F : M_1 \rightarrow M_2$ by

$$(1.8) \quad F(\exp_{p_1} tu_1) := \exp_{p_2}(tI(u_1)).$$

Note that F is a diffeomorphism between $M_1 \setminus \{q_1\}$ and $M_2 \setminus \{q_2\}$. Thanks to (1.6), we see that q_1 is a non-singular point of F , and hence F has no singular points on M_1 . By Theorem 1.3, for a sufficiently small $\eta > 0$, we can approximate F via smooth immersions $f_\eta : M_1 \rightarrow M_2$. By the topologies of M_1 and M_2 , we see that the local diffeomorphism f_η is a bijection. Therefore, M_1 and M_2 are diffeomorphic.

- The outline of the proof in the case where (1.7): Let $F : M_1 \rightarrow M_2$ denote the map defined by (1.8). And we embed M_2 into \mathbb{R}^m isometrically, where $m \geq n + 1$. By (1.7) and [21, Theorem 5.1], we can find $\delta > 0$ such that a locally smooth approximation $F_\varepsilon^{(q_1)}|_{B_\delta(q_1)}$ of F is an immersion into \mathbb{R}^m for all $\varepsilon \in (0, \delta)$, where $B_\delta(q_1) := \{x \in M_1 \mid d_{M_1}(q_1, x) < \delta\}$. Then, we have a local diffeomorphism $F_\varepsilon : M_1 \rightarrow \mathbb{R}^m$ defined by $F_\varepsilon := (1 - \varphi)F + \varphi F_\varepsilon^{(q_1)}$. Here, φ denotes a smooth function on M_1 satisfying $0 \leq \varphi \leq 1$ on M_1 , $\varphi \equiv 1$ on $B_r(q_1)$, and $\text{supp } \varphi \subset B_R(q_1)$, where $0 < r < R < \delta$. Choose a sufficiently small open neighborhood U of M_2 in \mathbb{R}^m so that the smooth (locally) distance projection $\pi : U \rightarrow M_2$ is well-defined. Since $F_\varepsilon(M_1) \subset U$ for any sufficiently small $\varepsilon > 0$, a map $f_\varepsilon := \pi \circ F_\varepsilon : M_1 \rightarrow M_2$ is a local diffeomorphism. Since f_ε is bijective, M_1 and M_2 are diffeomorphic. The fundamental course in the proof of Theorem 1.3 has a similar construction of f_ε , where we use the partition of unity.

Moreover, let $\bar{c} : [0, \pi] \rightarrow T_{q_2}M_2$ be a smooth curve defined by

$$(1.9) \quad \bar{c}(t) := c(0) \cos t + \dot{c}(0) \sin t,$$

where c is the curve in $\mathbb{S}_{q_2}^{n-1}$ defined by (1.5) and we set $\dot{c}(t) := dc/dt$. Note that $\|\dot{c}(0)\| \neq 1$ is possible. Then, we can replace (1.6) with the following condition (1.10):

Differentiable Exotic Sphere Theorem II. *If σ satisfies*

$$(1.10) \quad \angle(\bar{c}(t), c(t)) < \frac{\pi}{2}$$

for all geodesic segments $\gamma([0, \pi]) \subset \mathbb{S}_{q_1}^{n-1}$ with $\|\dot{\gamma}\| \equiv 1$, then M_1 and M_2 are diffeomorphic.

Remark 1.13 We find (1.10) in the process for proving lemmas that we need for the proof of the differentiable exotic sphere theorem I in the case where (1.6). By (1.10), q_1 is non-singular of the map F defined by (1.8), and hence F has no singular points on M_1 . By Theorem 1.3, we get the assertion.

Next, we will state the third application of Theorem 1.3 and its corollaries: Let M be a compact Riemannian manifold of dimension n . For any two distinct points $p, q \in M$, we set

$$D(p) := \{x \in M \mid d(p, x) < d(q, x)\}, \quad D(q) := \{x \in M \mid d(p, x) > d(q, x)\},$$

and

$$E_{p,q} := \{x \in M \mid d(p, x) = d(q, x)\},$$

where d denotes the distance function of M . Moreover, for a point $x \in M$, we set

$$\text{Crit}(x) := \{y \in M \mid o \in \partial d_x(y)\}.$$

With these notations, we have

Differentiable Twisted Sphere Theorem. *Take any two distinct points $p, q \in M$. If*

$$(1.11) \quad D(p) \cap \text{Crit}(p) = \{p\}, \quad D(q) \cap \text{Crit}(q) = \{q\},$$

and if for any geodesic segments α and β emanating from each point $x \in E_{p,q}$ to p and q , respectively,

$$(1.12) \quad \angle(\dot{\alpha}(0), \dot{\beta}(0)) > \frac{\pi}{2}$$

holds at x , then

(T-1) *M is a twisted sphere, and*

(T-2) *there exists a bi-Lipschitz homeomorphism between M and $S^n(1)$ which is a diffeomorphism except for the point q , where $S^n(1) := \{v \in \mathbb{R}^{n+1} \mid \|v\| = 1\}$.*

Furthermore, we have that

(T-3) *there exists a diffeomorphism $\sigma_q^p : \mathbb{S}_p^{n-1} \longrightarrow \mathbb{S}_q^{n-1}$ defined similar to (1.2) such that if the following condition (a) or (b) is satisfied, then M and $S^n(1)$ are diffeomorphic:*

(a) *$\text{Lip}^b((\sigma_q^p)^{-1})$ and $\|\ddot{c}\|$ satisfy (1.6), or if $(\sigma_q^p)^{-1}$ and n satisfy (1.7);*

(b) *$(\sigma_q^p)^{-1}$ satisfies (1.10).*

Here, $c : [0, \pi] \longrightarrow \mathbb{S}_S^{n-1}$ denotes a curve defined by (1.5) for $\sigma_{p_1}^{q_1} = (\sigma_q^p)^{-1}$, $\sigma_{q_2}^{p_2} = \sigma_S^N$, and all geodesic segments $\gamma([0, \pi]) \subset \mathbb{S}_q^{n-1}$ with $\|\dot{\gamma}\| \equiv 1$, where $N := (0, 0, \dots, 1)$, $S := (0, 0, \dots, -1) \in S^n(1)$. Note that the curve $\bar{c} : [0, \pi] \longrightarrow T_S S^n(1)$ in (1.10) is defined by (1.9) for this c .

Remark 1.14 We give two remarks on the differentiable twisted sphere theorem and several related topics for it:

- Since M is twisted by (T-1), M admits a metric such that the cut locus of some point in M is a single point by Weinstein's deformation technique for metrics. See Remark 4.3 for details.
- The diffeomorphism σ_q^p in (T-3) induces a boundary diffeomorphism $h_{\sigma_q^p} : S^{n-1} \rightarrow S^{n-1}$ such that $M = D^n \cup_{h_{\sigma_q^p}} D^n$, where D^n denotes the standard n -disc and $S^{n-1} = \partial D^n$ (See Sect. 4). Thus, by taking a contrapositive of (T-3), we could define clearly a boundary diffeomorphism to get an exotic sphere Σ^n ($n > 4$) from a twisted one. The fact that there are few explicit examples of such maps is the worthy of note. E.g., Durán's boundary diffeomorphism for an Σ^7 in [10].
- Donaldson and Sullivan [9] proved that there are smooth 4-manifolds which are homeomorphic, but not bi-Lipschitz.
- Let M be a piecewise linear (PL) n -manifold² of dimension $n \geq 5$ which has the homotopy type of S^n . Then, the Stallings-Zeeman theorem ([36] together with [39]) says that $M \setminus \{\text{point}\}$ is PL-homeomorphic to \mathbb{R}^n .
- For any closed Riemannian manifold M of dimension $n \geq 2$, Cheeger and Colding [4] found a positive number $\delta(n)$ depending on n such that if $\text{Ric}_M \geq n - 1$ and $\text{vol}(M) > \text{vol}(S^n(1)) - \delta(n)$, then M is diffeomorphic to $S^n(1)$, where Ric_M denotes Ricci curvature of M , and $\text{vol}(M)$ denotes the volume of M . Note that their result is a generalization of the pioneering work of Otsu-Shiohama-Yamaguchi [27] in the sectional curvature case.

Next, we will state three corollaries of the differentiable twisted sphere theorem: Applying the same argument in the proof of (T-2) to any twisted sphere, we have

Corollary 1.15 *For any twisted sphere Σ^n of general dimension n , there exists a bi-Lipschitz homeomorphism between Σ^n and $S^n(1)$ which is a diffeomorphism except for a single point. In particular, if $n > 4$, then there exists such a map between each exotic sphere Σ^n and $S^n(1)$.*

Remark 1.16 Let D^n be the standard n -disc and $S^{n-1} = \partial D^n$. Combining with results of Munkres [26], Kervaire-Milnor [22], and Cerf [3]³, we see that if $n \leq 6$, then the group $\Gamma_n := \text{Diff}(S^{n-1})/\text{Diff}(D^n)$ is trivial. Here, $\text{Diff}(X)$ denotes the topological group of orientation preserving diffeomorphisms of a smooth manifold X . Hence, every twisted n -sphere of dimension $n \leq 6$ is diffeomorphic to $S^n(1)$.

Since $\Gamma_4 = 0$ as in the above, there is no standard sphere but a twisted sphere. Hence, if an exotic 4-sphere Σ^4 exists, then Σ^4 is not twisted. Thus, we have

²A PL n -manifold is, by definition, a polyhedron admitting a linear triangulation that satisfies that the link of each vertex is combinatorially equivalent to the boundary of the n -simplex.

³An alternative proof of $\Gamma_4 = 0$ is found in Eliashberg's [11].

Corollary 1.17 *If Σ^4 exists, then Σ^4 does not satisfy (1.11), or (1.12).*

Remark 1.18 So far many topological 4-spheres regarded as exotic are indeed standard. E.g., Akbulut's [1] and Gompf's [13]. We refer to [12] about the difficulty to solve SPC4. We can find an interesting work on the relationship between Stein 4-manifolds and SPC4 in Yasui's [38].

Let M be a Grove-Shiohama type n -sphere, and let $p, q \in M$ such that $d(p, q) = \text{diam}(M)$, where d denotes the distance function of M . Then, Toponogov's comparison theorem shows the following process: First $M \setminus B_{\pi/2}(p)$ is convex, where $B_{\pi/2}(p) := \{x \in M \mid d(p, x) < \pi/2\}$, second q is a unique point, and finally $\text{Crit}(p) = \{p, q\}$ (See [18], or [23], [24] for details). By the same process above, we have $\text{Crit}(q) = \{p, q\}$ since $K_M \geq 1$ everywhere. Thus, M satisfies (1.11) and (1.12). By the differentiable twisted sphere theorem, we have

Corollary 1.19 *Let M be a Grove-Shiohama type n -sphere, and let $p, q \in M$ such that $d(p, q) = \text{diam}(M)$. Then, there exists a bi-Lipschitz homeomorphism between M and $S^n(1)$ which is a diffeomorphism except for q .*

Remark 1.20 We give a remark on Corollary 1.19 and two related results for it:

- Since any Grove-Shiohama type n -sphere M is twisted, M of dimension $n \leq 6$ is diffeomorphic to $S^n(1)$. See Remark 1.16 above.
- Grove and Wilhelm [19] proved that every Grove-Shiohama type n -sphere is diffeomorphic to $S^n(1)$ if the diffeomorphism stability question is “yes” in their sense.
- Petersen and Wilhelm [28] announced that the Gromoll-Meyer exotic sphere in [14] admits a metric with positive sectional curvature everywhere.

The organization of this article is as follows: Sect. 2 has three subsections. In Sect. 2.1, we study an approximation of a Lipschitz function on a Riemannian manifold M . Using non-smooth analysis, we prove, as a main lemma (Lemma 2.14), that (1.12) also holds for two gradient vector fields on a compact set in M of smooth approximations of two distance functions to distinct points in M . In Sect. 2.2, we establish an approximation method for a Lipschitz map from a compact Riemannian manifold into a Riemannian manifold using some techniques from non-smooth analysis and the partition of unity, and prove our main theorem, Theorem 1.3, applying the method. In Sect. 2.3, we prove Corollary 1.10 applying Theorem 1.3. Sect. 3 has four subsections. In the first two subsections 3.1 and 3.2, we give preliminaries to proofs of the differentiable exotic sphere theorems I and II in assuming (1.6), (1.7), and (1.10), respectively. In Sect. 3.3, we prove the two theorems. In Sect. 3.4, we discuss why we need the restriction of \ddot{c} in (1.6) for the two theorems. In Sect. 4, we prove the differentiable twisted sphere theorem applying Lemma 2.14 and Theorem 1.3.

2 Approximations of Lipschitz maps via immersions

We prove here our main theorem, Theorem 1.3. For the proof, we establish an approximation method for a Lipschitz map from a compact Riemannian manifold into a Riemannian manifold using non-smooth analysis. As was mentioned in Remark 1.4, this approximation method generalizes the whole method of Grove-Shiohama [18]. Note that we do not assume curvature assumptions at all, and that the smoothing technique bases on the partition of unity, while that in [18], [21] depend on the center of mass technique constructed by Grove and Karcher [17].

2.1 Approximations of Lipschitz functions

We first treat an approximation of a Lipschitz function f on a Riemannian manifold M of dimension n . As a main lemma (Lemma 2.14) in this subsection, we prove that our approximation method keeps (1.12) for two gradient vector fields $\nabla(d_p)_\varepsilon, \nabla(d_q)_\varepsilon$ on a compact set in M of smooth approximations $(d_p)_\varepsilon, (d_q)_\varepsilon$ of two distance functions d_p, d_q to distinct points $p, q \in M$, where d denotes the distance function of M . This lemma will be applied to the proof of the differentiable twisted sphere theorem in Sect. 4.

Take any point $p \in M$, and fix it. Since the exponential map \exp_p on $T_p M$ is a diffeomorphism from $\mathbb{B}_r(o_p)$ onto $B_r(p)$ for a sufficiently small $r > 0$, we denote by \exp_p^{-1} the inverse map of $\exp_p|_{\mathbb{B}_r(o_p)}$. Here we set

$$\mathbb{B}_r(o_p) := \{v \in T_p M \mid \|v\| < r\}, \quad B_r(p) := \{q \in M \mid d(p, q) < r\},$$

where o_p denotes the origin of $T_p M$. In what follows, we identify $T_p M$ with Euclidean n -dimensional space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. Then, we may define a smooth approximation of the f around p :

Definition 2.1 For each $\varepsilon > 0$, let $f_\varepsilon^{(p)} : B_r(p) \rightarrow \mathbb{R}$ denote the function defined by

$$f_\varepsilon^{(p)}(q) := \int_{\mathbb{R}^n} f(q(y)) \rho_\varepsilon(y) dy = \int_{\mathbb{R}^n} f(\exp_p(y)) \rho_\varepsilon(\exp_p^{-1} q - y) dy,$$

where we set $q(y) := \exp_p(\exp_p^{-1} q - y)$, and the function ρ_ε denotes the mollifier. Refer to [31], [21], or [18] for details of the mollifier.

Let $\{\psi_i\}_{i=1}^\infty$ denote the partition of unity subordinate to a locally finite cover $\{B_{r_i}(p_i)\}$ of strongly convex balls of M . Define a smooth approximation f_ε of the f on M by

$$f_\varepsilon(q) := \sum_{i=1}^\infty \psi_i(q) f_\varepsilon^{(p_i)}(q).$$

By definition,

$$(2.1) \quad f_\varepsilon^{(p_i)}(q) = \int_{\mathbb{R}^n} f(q_i(y)) \rho_\varepsilon(y) dy$$

for $q \in B_{r_i}(p_i)$, where

$$(2.2) \quad q_i(y) := \exp_{p_i}(\exp_{p_i}^{-1} q - y).$$

Lemma 2.2 For each center p_i of $B_{r_i}(p_i)$,

$$(2.3) \quad \sup_{q \in \text{supp } \psi_i} |f_\varepsilon^{(p_i)}(q) - f(q)| \leq \varepsilon \cdot \text{Lip}(f) \cdot \text{Lip}(\exp_{p_i} |_{\mathbb{B}_{r_i}(o_{p_i})})$$

holds for all $\varepsilon \in (0, \varepsilon_i)$, where $\varepsilon_i := r_i - \max\{\|\exp_{p_i}^{-1} q\| \mid q \in \text{supp } \psi_i\}$, and $\text{Lip}(f)$ and $\text{Lip}(\exp_{p_i} |_{\mathbb{B}_{r_i}(o_{p_i})})$ denotes the Lipschitz constants of f and $\exp_p |_{\mathbb{B}_{r_i}(o_{p_i})}$, respectively, i.e.,

$$(2.4) \quad \text{Lip}(f) := \sup \left\{ \frac{|f(q_1) - f(q_2)|}{d(q_1, q_2)} \mid q_1, q_2 \in M, q_1 \neq q_2 \right\}$$

and

$$\text{Lip}(\exp_{p_i} |_{\mathbb{B}_{r_i}(o_{p_i})}) := \sup \left\{ \frac{d(\exp_{p_i} v, \exp_{p_i} w)}{\|v - w\|} \mid v, w \in \mathbb{B}_{r_i}(o_{p_i}), v \neq w \right\}.$$

Proof. Take any point $q \in \text{supp } \psi_i$. Then, for any $y \in \mathbb{B}_{\varepsilon_i}(o_{p_i})$, we have, by the triangle inequality,

$$\|\exp_{p_i}^{-1} q - y\| \leq \|\exp_{p_i}^{-1} q\| + \|y\| < r_i.$$

Thus, $\exp_{p_i}^{-1} q - y$ and $\exp_{p_i}^{-1} q$ are elements of $\mathbb{B}_{r_i}(o_{p_i})$ for all $y \in \mathbb{B}_{\varepsilon_i}(o_{p_i})$. Hence,

$$(2.5) \quad d(q_i(y), q) \leq \text{Lip}(\exp_{p_i} |_{\mathbb{B}_{r_i}(o_{p_i})}) \|\exp_{p_i}^{-1} q - y\| = \text{Lip}(\exp_{p_i} |_{\mathbb{B}_{r_i}(o_{p_i})}) \|y\|.$$

On the other hand, since $\int_{\mathbb{R}^n} \rho_\varepsilon(y) dy = 1$, we have $f(q) = \int_{\|y\| < \varepsilon} f(q) \rho_\varepsilon(y) dy$. Then,

$$(2.6) \quad \begin{aligned} |f_\varepsilon^{(p_i)}(q) - f(q)| &\leq \int_{\|y\| < \varepsilon} |f(q_i(y)) - f(q)| \rho_\varepsilon(y) dy \\ &\leq \text{Lip}(f) \int_{\|y\| < \varepsilon} d(q_i(y), q) \rho_\varepsilon(y) dy, \end{aligned}$$

where note that $\|y\| < \varepsilon$ with $\rho_\varepsilon(y) \neq 0$. Combining (2.5) with (2.6), we get (2.3). \square

Take any $p \in M$ and any p_i with $p \in B_{r_i}(p_i)$, and fix them in the following. Moreover, we assume $q \in \text{supp } \psi_i$ in this situation.

Lemma 2.3 For any $\tilde{u} \in \mathbb{S}_q^{n-1} := \{v \in T_q M \mid \|v\| = 1\}$,

$$(2.7) \quad (df_\varepsilon^{(p_i)})_q(\tilde{u}) = \int_{\mathbb{R}^n} df_{q_i(y)}(Y_y^{(\tilde{u})}(1)) \rho_\varepsilon(y) dy$$

holds, where $q_i(y) \in M$ is by definition (2.2), and

$$Y_y^{(\tilde{u})}(t) := \frac{\partial}{\partial s} \exp_{p_i} t(\exp_{p_i}^{-1}(\exp_q s \tilde{u}) - y) \Big|_{s=0}$$

is a Jacobi field along the geodesic $\exp_{p_i} t(\exp_{p_i}^{-1} q - y)$ for each y .

Proof. Since

$$(df_\varepsilon^{(p_i)})_q(\tilde{u}) = \frac{d}{ds} f_\varepsilon^{(p_i)}(\exp_q s \tilde{u}) \Big|_{s=0},$$

we get (2.7) by (2.1). \square

For a pair of points q_1 and q_2 of M admitting a unique minimal geodesic segment γ , let $\tau_{q_2}^{q_1} : T_{q_1}M \rightarrow T_{q_2}M$ denote the parallel transportation along γ . In what follows, we will omit brackets of $\tau_{q_2}^{q_1}(u)$ ($u \in T_{q_1}M$) for simplicity, i.e, $\tau_{q_2}^{q_1}u := \tau_{q_2}^{q_1}(u)$.

Lemma 2.4 *Let $\tilde{U}_{q_i(y)} := \tau_{q_i(y)}^q \tilde{u}$. Then, we have*

$$\left| (df_\varepsilon^{(p_i)})_q(\tilde{u}) - \int_{\mathbb{R}^n} df_{q_i(y)}(\tilde{U}_{q_i(y)}) \rho_\varepsilon(y) dy \right| \leq \text{Lip}(f) \sup_{y \in \mathbb{B}_\varepsilon(o_{p_i})} \|Y_y^{(\tilde{u})}(1) - \tilde{U}_{q_i(y)}\|.$$

Proof. By Lemma 2.3 and (2.4), we get

$$\begin{aligned} \left| (df_\varepsilon^{(p_i)})_q(\tilde{u}) - \int_{\mathbb{R}^n} df_{q_i(y)}(\tilde{U}_{q_i(y)}) \rho_\varepsilon(y) dy \right| &\leq \int_{\mathbb{R}^n} |df_{q_i(y)}(Y_y^{(\tilde{u})}(1) - \tilde{U}_{q_i(y)})| \rho_\varepsilon(y) dy \\ &\leq \text{Lip}(f) \sup_{y \in \mathbb{B}_\varepsilon(o_{p_i})} \|Y_y^{(\tilde{u})}(1) - \tilde{U}_{q_i(y)}\|. \end{aligned}$$

□

Lemma 2.5 *For any $\eta > 0$, there exists a number $\varepsilon_i(\eta) > 0$ such that*

$$\sup\{\|Y_y^{(\tilde{u})}(1) - \tilde{U}_{q_i(y)}\| \mid y \in \mathbb{B}_{\varepsilon_i(\eta)}(o_{p_i}), q \in \text{supp } \psi_i, \tilde{u} \in \mathbb{S}_q^{n-1}\} < \eta.$$

Proof. Let $\eta > 0$ be an arbitrary number. Take any $q \in \text{supp } \psi_i$ and any $\tilde{u} \in \mathbb{S}_q^{n-1}$. Since $Y_y^{(\tilde{u})}(1) = \tilde{U}_{q_i(y)} = \tilde{u}$ for $y = o$, there exists a positive number $\varepsilon(q, \tilde{u}, p_i, \eta) > 0$ such that $\|Y_y^{(\tilde{u})}(1) - \tilde{U}_{q_i(y)}\| < \eta$ for all $y \in \mathbb{B}_{\varepsilon(q, \tilde{u}, p_i, \eta)}(o_{p_i})$. Since $\text{supp } \psi_i$ and \mathbb{S}_q^{n-1} are compact, there exists a positive number $\varepsilon_i(\eta)$ such that $\|Y_y^{(\tilde{u})}(1) - \tilde{U}_{q_i(y)}\| < \eta$ for all $q \in \text{supp } \psi_i$, $\tilde{u} \in \mathbb{S}_q^{n-1}$, and $y \in \mathbb{B}_{\varepsilon_i(\eta)}(o_{p_i})$. □

For each $v \in \mathbb{R}^n = T_p M$, let $\pi_p(v)$ be the nearest point on $\partial f(p)$ from v . Since $\partial f(p)$ is convex, the nearest point is uniquely determined. Hence, it is easy to check that the map $\pi_p : \mathbb{R}^n \rightarrow \partial f(p)$ is continuous.

Lemma 2.6 *Set*

$$(2.8) \quad v_\varepsilon^{(i)} := \int_{\mathbb{R}^n} \pi_p(\tau_p^q \tau_q^{q_i(y)} \nabla f(q_i(y))) \rho_\varepsilon(y) dy$$

and $u := \tau_p^q \tilde{u}$. Then, we have

$$\left| \int_{\mathbb{R}^n} df_{q_i(y)}(\tilde{U}_{q_i(y)}) \rho_\varepsilon(y) dy - \langle v_\varepsilon^{(i)}, u \rangle \right| \leq \int_{\mathbb{R}^n} \|(1 - \pi_p)(\tau_p^q \tau_q^{q_i(y)} \nabla f(q_i(y)))\| \rho_\varepsilon(y) dy,$$

where we set $\tilde{U}_{q_i(y)} := \tau_{q_i(y)}^q \tilde{u}$.

Proof. It is easy to check that $df_{q_i(y)}(\tilde{U}_{q_i(y)}) = \langle \tau_p^q \tau_q^{q_i(y)} \nabla f(q_i(y)), u \rangle$. Thus, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^n} df_{q_i(y)}(\tilde{U}_{q_i(y)}) \rho_\varepsilon(y) dy - \langle v_\varepsilon^{(i)}, u \rangle \right| &= \left| \int_{\mathbb{R}^n} \langle (1 - \pi_p)(\tau_p^q \tau_q^{q_i(y)} \nabla f(q_i(y))), u \rangle \rho_\varepsilon(y) dy \right| \\ &\leq \int_{\mathbb{R}^n} \|(1 - \pi_p)(\tau_p^q \tau_q^{q_i(y)} \nabla f(q_i(y)))\| \rho_\varepsilon(y) dy. \end{aligned}$$

□

Definition 2.7 A map $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is called a *locally L^1 -map* if each $F^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a locally L^1 -function, where $F^i(x)$ denotes the i -th component of $F(x) \in \mathbb{R}^k$.

Lemma 2.8 Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a locally L^1 -map, and $C \subset \mathbb{R}^k$ a compact convex set. If $F(\mathbb{R}^n) \subset C$, then $\int_{\mathbb{R}^n} \rho(x)F(x)dx \in C$ for all non-negative continuous function $\rho : \mathbb{R}^k \longrightarrow \mathbb{R}$ with $\int_{\mathbb{R}^n} \rho(x)dx = 1$.

Proof. Let $\{H_\alpha\}_{\alpha \in \Gamma}$ be the family of all closed half spaces in \mathbb{R}^k . It is well known that $K = \bigcap_{K \subset H_\alpha} H_\alpha$ for any closed convex subset K of \mathbb{R}^k . Choose any closed half space H_α with $C \subset H_\alpha$. Let n_α denote the inward pointing unit normal vector of H_α and $v_\alpha \in \mathbb{R}^k$ the nearest point of C from ∂H_α . Then, $\langle F(x) - v_\alpha, n_\alpha \rangle \geq 0$ holds for all x , since $F(x) \in C$. Since $\int_{\mathbb{R}^n} \rho(x)dx = 1$, $\int_{\mathbb{R}^n} \rho(x)v_\alpha dx = v_\alpha$. Thus,

$$\left\langle \int \rho(x)F(x)dx - v_\alpha, n_\alpha \right\rangle = \int \rho(x)\langle F(x) - v_\alpha, n_\alpha \rangle dx \geq 0,$$

and hence $\int_{\mathbb{R}^n} \rho(x)F(x)dx \in H_\alpha$ for all H_α with $C \subset H_\alpha$. Thus, $\int_{\mathbb{R}^n} \rho(x)F(x)dx \in C$. \square

Lemma 2.9 For any $\eta > 0$, there exists a number $\delta(p, \eta) > 0$ such that

$$\tau_p^{q_1} \tau_{q_1}^{q_2} \nabla f(q_2) \in \partial f(p)_\eta := \bigcup_{v \in \partial f(p)} \mathbb{B}_\eta(v)$$

holds for all $q_1 \in B_{\delta(p, \eta)}(p)$ and $q_2 \in B_{\delta(p, \eta)}(p) \cap (M \setminus E_f)$.

Proof. Take any two sequences $\{p_i\}$ of $p_i \in M$ and $\{q_i\}$ of $q_i \in M \setminus E_f$ both of which are convergent to the point p . If the limit $\lim_{i \rightarrow \infty} \nabla f(q_i)$ exists, then $\lim_{i \rightarrow \infty} \tau_p^{p_i} \tau_{p_i}^{q_i} \nabla f(q_i) = \lim_{i \rightarrow \infty} \nabla f(q_i) \in \partial f(p)$. This implies the existence of the positive number $\delta(p, \eta)$. \square

Lemma 2.10 For any $\eta > 0$, there exist numbers $\delta_1(p, \eta) > 0$ and $\varepsilon(p, p_i, \eta) > 0$ such that

$$\tau_p^q \nabla f_\varepsilon^{(p_i)}(q) \in \partial f(p)_\eta$$

for all $q \in B_{\delta_1(p, \eta)}(p) \cap \text{supp } \psi_i$, and for all $\varepsilon \in (0, \varepsilon(p, p_i, \eta))$.

Proof. It follows from Lemmas 2.4 and 2.5 that

$$(2.9) \quad \left| (df_\varepsilon^{(p_i)})_q(\tilde{u}) - \int_{\mathbb{R}^n} df_{q_i(y)}(\tilde{U}_{q_i(y)}) \rho_\varepsilon(y) dy \right| < \frac{\eta}{2}$$

for all $\varepsilon \in (0, \varepsilon_i(\eta/2\text{Lip}(f)))$, $q \in \text{supp } \psi_i$, and $\tilde{u} \in \mathbb{S}_q^{n-1}$. If ε is not greater than

$$\min \left\{ r_i - \max_{q \in \text{supp } \psi_i} \|\exp_{p_i}^{-1} q\|, \frac{\delta(p, \eta/2)}{2 \text{Lip}(\exp_{p_i}|_{\mathbb{B}_{r_i}(o_{p_i})})} \right\},$$

then $\|\exp_{p_i}^{-1} q - y\| \leq \|\exp_{p_i}^{-1} q\| + \|y\| < r_i$ for all $y \in \mathbb{B}_\varepsilon(o_{p_i})$. Hence,

$$d(q_i(y), q) \leq \text{Lip}(\exp_{p_i}|_{\mathbb{B}_{r_i}(o_{p_i})})\|y\| < \frac{\delta(p, \eta/2)}{2}$$

for all $y \in \mathbb{B}_\varepsilon(o_{p_i})$. Thus, we have

$$d(p, q_i(y)) \leq d(p, q) + d(q, q_i(y)) < \frac{\delta(p, \eta/2)}{2} + \frac{\delta(p, \eta/2)}{2} = \delta(p, \eta/2)$$

for all $q \in \text{supp } \psi_i \cap B_{\delta_1(p, \eta)}(p)$, where we set $\delta_1(p, \eta) := \delta(p, \eta/2)/2$. Let

$$\varepsilon(p, p_i, \eta) := \min \left\{ \varepsilon_i(\eta/2 \text{Lip}(f)), \frac{\delta_1(p, \eta)}{\text{Lip}(\exp_{p_i}|_{\mathbb{B}_{r_i}(o_{p_i})})}, r_i - \max_{q \in \text{supp } \psi_i} \|\exp_{p_i}^{-1} q\| \right\}.$$

From Lemma 2.9, we get

$$\|(1 - \pi_p)(\tau_p^q \tau_q^{q_i(y)} \nabla f(q_i(y)))\| < \frac{\eta}{2}$$

for all $q \in B_{\delta_1(p, \eta)}(p)$ and almost all $y \in \mathbb{B}_{\varepsilon(p, p_i, \eta)}(o_{p_i})$. Therefore, by Lemma 2.6, we obtain

$$(2.10) \quad \left| \int_{\mathbb{R}^n} df_{q_i(y)}(\tilde{U}_{q_i(y)}) \rho_\varepsilon(y) dy - \langle v_\varepsilon^{(i)}, u \rangle \right| < \frac{\eta}{2}$$

for all $q \in B_{\delta_1(p, \eta)}(p)$ and all $\varepsilon \in (0, \varepsilon(p, p_i, \eta))$. By the triangle inequality and the equations (2.9), (2.10), we obtain

$$(2.11) \quad |(df_\varepsilon^{(p_i)})_q(\tilde{u}) - \langle v_\varepsilon^{(i)}, u \rangle| < \eta$$

for all $\tilde{u} \in \mathbb{S}_q^{n-1}$, $q \in \text{supp } \psi_i \cap B_{\delta_1(p, \eta)}(p)$, and $\varepsilon \in (0, \varepsilon(p, p_i, \eta))$. Since $(df_\varepsilon^{(p_i)})_q(\tilde{u}) = \langle \tau_p^q \nabla f_\varepsilon^{(p_i)}(q), u \rangle$ and u is arbitrarily chosen, we have, by (2.11),

$$\|\tau_p^q \nabla f_\varepsilon^{(p_i)}(q) - v_\varepsilon^{(i)}\| = \frac{|\langle \tau_p^q \nabla f_\varepsilon^{(p_i)}(q) - v_\varepsilon^{(i)}, \tau_p^q \nabla f_\varepsilon^{(p_i)}(q) - v_\varepsilon^{(i)} \rangle|}{\|\tau_p^q \nabla f_\varepsilon^{(p_i)}(q) - v_\varepsilon^{(i)}\|} < \eta$$

for all $q \in \text{supp } \psi_i \cap B_{\delta_1(p, \eta)}(p)$ and $\varepsilon \in (0, \varepsilon(p, p_i, \eta))$. By Lemma 2.8 and (2.8), $v_\varepsilon^{(i)} \in \partial f(p)$. Hence, $\tau_p^q \nabla f_\varepsilon^{(p_i)}(q) \in \partial f(p)_\eta$ for all $q \in \text{supp } \psi_i \cap B_{\delta_1(p, \eta)}(p)$, and $\varepsilon \in (0, \varepsilon(p, p_i, \eta))$. \square

Lemma 2.11 *For any $\eta > 0$, there exist numbers $\delta_2(p, \eta) > 0$ and $\varepsilon(p, \eta) > 0$ such that*

$$\tau_p^q \nabla f_\varepsilon(q) \in \partial f(p)_\eta$$

for all $q \in B_{\delta_2(p, \eta)}(p)$ and all $\varepsilon \in (0, \varepsilon(p, \eta))$.

Proof. By Lemma 2.10, $\tau_p^q \nabla f_\varepsilon^{(p_i)}(q) \in \partial f(p)_{\eta/2}$ holds for all $q \in B_{\delta_1(p, \eta/2)}(p) \cap \text{supp } \psi_i$ and $\varepsilon \in (0, \varepsilon(p, p_i, \eta/2))$. Since $\partial f(p)_{\eta/2}$ is convex,

$$(2.12) \quad \sum_i \psi_i(q) \cdot \tau_p^q \nabla f_\varepsilon^{(p_i)}(q) \in \partial f(p)_{\eta/2}$$

for all $q \in B_{\delta_1(p, \eta/2)}(p)$ and $\varepsilon \in (0, \varepsilon_1(p, \eta))$, where

$$\varepsilon_1(p, \eta) := \min\{\varepsilon(p, p_i, \eta/2) \mid B_{\delta_1(p, \eta/2)}(p) \cap \text{supp } \psi_i \neq \emptyset\}.$$

Fix any $q \in B_{\delta_2(p, \eta)}(p)$, where $\delta_2(p, \eta) := \delta_1(p, \eta/2)$, and take any $\tilde{u} \in \mathbb{S}_q^{n-1}$. Since $\sum_i \psi_i = 1$, $\sum_i d\psi_i(\tilde{u}) = 0$ holds. Then, we get

$$(2.13) \quad (df_\varepsilon)_q(\tilde{u}) = \sum_i \psi_i(q) df_\varepsilon^{(p_i)}(\tilde{u}) + \sum_i d\psi_i(\tilde{u})(f_\varepsilon^{(p_i)}(q) - f(q))$$

By Lemma 2.2,

$$\left| \sum_i d\psi_i(\tilde{u})(f_\varepsilon^{(p_i)}(q) - f(q)) \right| \leq \varepsilon \sum_i |d\psi_i(\tilde{u})| \cdot \text{Lip}(f) \cdot \text{Lip}(\exp_{p_i} \mid_{\mathbb{B}_{r_i}(o_{p_i})})$$

holds. Therefore, there exists a number $\varepsilon(p, \eta) \in (0, \varepsilon_1(p, \eta)]$ such that

$$(2.14) \quad \left| \sum_i d\psi_i(\tilde{u})(f_\varepsilon^{(p_i)}(q) - f(q)) \right| < \frac{\eta}{2}$$

for all $\varepsilon \in (0, \varepsilon(p, \eta))$ and for all $q \in B_{\delta_2(p, \eta)}(p)$. By (2.13) and (2.14),

$$\left| \langle \nabla f_\varepsilon(q), \tilde{u} \rangle - \left\langle \sum_i \psi_i(q) \nabla f_\varepsilon^{(p_i)}(q), \tilde{u} \right\rangle \right| < \frac{\eta}{2}.$$

If we set $u := \tau_p^q \tilde{u}$, then

$$\left| \langle \tau_p^q \nabla f_\varepsilon(q), u \rangle - \left\langle \sum_i \psi_i(q) \cdot \tau_p^q \nabla f_\varepsilon^{(p_i)}(q), u \right\rangle \right| < \frac{\eta}{2}.$$

Since u is any unit tangent vector,

$$\left\| \tau_p^q \nabla f_\varepsilon(q) - \sum_i \psi_i(q) \cdot \tau_p^q \nabla f_\varepsilon^{(p_i)}(q) \right\| < \frac{\eta}{2}.$$

Hence, from (2.12), it follows that $\tau_p^q \nabla f_\varepsilon(q) \in \partial f(p)_\eta$ for all $q \in B_{\delta_2(p, \eta)}(p)$ and all $\varepsilon \in (0, \varepsilon(p, \eta))$. \square

Lemma 2.12 *Let $A, B \subset \mathbb{R}^n$ be compact sets such that $\angle(u, v) > \pi/2$ holds for all $u \in A$ and $v \in B$. Then, $o \notin \text{Conv}(A)$ and $o \notin \text{Conv}(B)$. Moreover, $\angle(u, v) > \pi/2$ also holds for all $u \in \text{Conv}(A)$ and $v \in \text{Conv}(B)$.*

Proof. Take any $u \in A$, and fix it. Since $\angle(u, v) > \pi/2$ for all $v \in B$, B is a subset of the open convex cone $V_u := \{v \in \mathbb{R}^n \setminus \{o\} \mid \angle(u, v) > \pi/2\}$. Thus, $B \subset \bigcap_{u \in A} V_u$. The set $\bigcap_{u \in A} V_u$ is convex, since each V_u is convex. From the definition of the convex hull, $\text{Conv}(B) \subset \bigcap_{u \in A} V_u$. Hence, we have proved that $o \notin \text{Conv}(B)$, and that $\angle(u, v) > \pi/2$ for all $u \in A$ and all $v \in \text{Conv}(B)$. On the other hand, choose any $v \in \text{Conv}(B)$, and fix it. Since $\angle(u, v) > \pi/2$ for all $u \in A$, A is a subset of the open convex cone V_v . Then, $A \subset \bigcap_{v \in \text{Conv}(B)} V_v$, and hence $\text{Conv}(A) \subset \bigcap_{v \in \text{Conv}(B)} V_v$. In particular, $o \notin \text{Conv}(A)$ and $\angle(u, v) > \pi/2$ for all $u \in \text{Conv}(A)$ and all $v \in \text{Conv}(B)$. \square

Lemma 2.13 *Let f and h be Lipschitz functions on M , respectively. Assume that $o \notin \partial f(x)$, $o \notin \partial h(x)$ on a compact set $K \subset M$. If $\angle(u, v) > \pi/2$ holds for all $x \in K$ and all $u \in \partial f(x)$, $v \in \partial h(x)$, then there exists a number $\eta_K > 0$ such that, for any $x \in K$ and any $u \in \partial f(x)_{\eta_K}$, $v \in \partial h(x)_{\eta_K}$, $\angle(u, v) > \pi/2$ holds.*

Proof. By supposing that the conclusion is false, we will get a contradiction. Then, for each positive integer i , there exist a point $x_i \in K$, $u_i \in \partial f(x_i)$, and $v_i \in \partial h(x_i)$ such that $\angle(u_i, v_i) < \pi/2 + 1/i$. Since K is compact, we can assume, taking a subsequence, if necessary, that $x := \lim_{i \rightarrow \infty} x_i \in K$ exists. By definition, any limits of the sequences $\{\tau_x^{x_i} u_i\}$ and $\{\tau_x^{x_i} v_i\}$ are elements of $\partial f(x)$ and $\partial h(x)$, respectively. Since $\angle(\tau_x^{x_i} u_i, \tau_x^{x_i} v_i) = \angle(u_i, v_i) < \pi/2 + 1/i$ for each i , we get $\angle(u, v) \leq \pi/2$ for some $u \in \partial f(x)$ and $v \in \partial h(x)$. This is a contradiction. \square

Lemma 2.14 *Let $K \subset M$ be a compact set. Assume that there exist two distinct points $p, q \in M \setminus K$ such that for any minimal geodesic segments α and β emanating from each point $x \in K$ to p and q , respectively, $\angle(\dot{\alpha}(0), \dot{\beta}(0)) > \pi/2$ holds. Then, for any sufficiently small $\varepsilon > 0$, $\angle(\nabla(d_p)_\varepsilon, \nabla(d_q)_\varepsilon) > \pi/2$ holds on K , where $d_p(x) := d(p, x)$ for all $x \in M$.*

Proof. Let x be any point of K . It follows from Lemma 2.12 that $o \notin \partial d_p(x) \cup \partial d_q(x)$ and $\angle(u, v) > \pi/2$ holds for all $u \in \partial d_p(x)$ and $v \in \partial d_q(x)$. Then, by Lemma 2.13, there exists a positive number η_K such that, for any $x \in K$, $u \in \partial d_p(x)_{\eta_K}$, and $v \in \partial d_q(x)_{\eta_K}$, $\angle(u, v) > \pi/2$ holds again. It follows from Lemma 2.11 that, for each point $x \in K$, there exist two numbers $\delta_2(x, \eta_K) > 0$ and $\varepsilon(x, \eta_K) > 0$ such that $\tau_x^{q_1} \nabla(d_p)_\varepsilon(q_1) \in \partial d_p(x)_{\eta_K}$ and $\tau_x^{q_1} \nabla(d_q)_\varepsilon(q_1) \in \partial d_q(x)_{\eta_K}$ for all $q_1 \in B_{\delta_2(x, \eta_K)}(x)$ and $\varepsilon \in (0, \varepsilon(x, \eta_K))$. Since $\angle(\nabla(d_p)_\varepsilon(q_1), \nabla(d_q)_\varepsilon(q_1)) = \angle(\tau_x^{q_1} \nabla(d_p)_\varepsilon(q_1), \tau_x^{q_1} \nabla(d_q)_\varepsilon(q_1))$, it follows from Lemma 2.13 that $\angle(\nabla(d_p)_\varepsilon(q_1), \nabla(d_q)_\varepsilon(q_1)) > \pi/2$ for all $q_1 \in B_{\delta_2(x, \eta_K)}(x)$ and all $\varepsilon \in (0, \varepsilon(x, \eta_K))$. This implies that, for any sufficiently small $\varepsilon > 0$, $\angle(\nabla(d_p)_\varepsilon, \nabla(d_q)_\varepsilon) > \pi/2$ holds on K since K is compact. \square

2.2 Approximations of Lipschitz maps: Proof of Theorem 1.3

In this subsection, using some techniques from non-smooth analysis and the partition of unity, we first treat an approximation of a Lipschitz map F from a compact Riemannian manifold M of dimension n into a Riemannian manifold N of dimension k , and finally prove Theorem 1.3 applying the approximation method.

Let N be embedded into Euclidean m -dimensional space $(\mathbb{R}^m, \langle \cdot, \cdot \rangle)$, where $m \geq k+1$. We may assume that N is isometrically embedded into \mathbb{R}^m by introducing the induced metric from the space. Here, note that the notion of the singular point of a Lipschitz map is independent of the choice of the Riemannian metric (See (1.1) and Definition 1.5). The Lipschitz map F is therefore a map from M into \mathbb{R}^m . Then, we may define a smooth approximation of F on a convex ball $B_r(p)$ of radius r , centered at each point $p \in M$.

Definition 2.15 For each $\varepsilon > 0$, let $F_\varepsilon^{(p)} : B_r(p) \longrightarrow \mathbb{R}^m$ denote the map defined by

$$(2.15) \quad F_\varepsilon^{(p)}(q) := \int_{\mathbb{R}^n} \rho_\varepsilon(y) F(q(y)) dy = \int_{\mathbb{R}^n} \rho_\varepsilon(\exp_p^{-1} q - y) F(\exp_p(y)) dy,$$

where $q(y) := \exp_p(\exp_p^{-1} q - y)$ and ρ_ε denotes the mollifier.

Lemma 2.16 *For any $\varepsilon > 0$ and any $q \in B_r(p)$,*

$$\|F_\varepsilon^{(p)}(q) - F(q)\| \leq \varepsilon \cdot \text{Lip}(F) \cdot \text{Lip}(\exp_p|_{\mathbb{B}_{r+\varepsilon}(o_p)})$$

holds, where $\|\cdot\|$ denotes the Euclidean norm of \mathbb{R}^m .

Proof. After the fashion of the proof of Lemma 2.2, we have the desired inequality. \square

Since M is compact, we can choose finitely many convex balls $B_{r_i}(p_i)$, $i = 1, 2, \dots, \ell$, which cover M . Take a partition of unity φ_i subordinate to $\{B_{r_i}(p_i)\}$ so that $\text{supp } \varphi_i \subset B_{r_i}(p_i)$ for each i . Then, for each $\varepsilon > 0$, we define a global approximation F_ε of F by

$$(2.16) \quad F_\varepsilon(q) = \sum_{i=1}^{\ell} \varphi_i(q) F_\varepsilon^{(p_i)}(q).$$

Lemma 2.17 *For any $\varepsilon > 0$ and any $q \in M$, we have*

$$\|F_\varepsilon(q) - F(q)\| \leq \varepsilon \cdot \text{Lip}(F) \sum_{i=1}^{\ell} \varphi_i(q) \text{Lip}(\exp_{p_i}|_{\mathbb{B}_{r_i+\varepsilon}(p_i)}).$$

Proof. Since $\sum_{i=1}^{\ell} \varphi_i = 1$, we get $F(q) = \sum_{i=1}^{\ell} \varphi_i(q) F(q)$. Hence, by Lemma 2.16 and the triangle inequality, we have the desired inequality. \square

In what follows, for a pair of points p and q of M or N admitting a unique minimal geodesic segment γ , we denote by τ_q^p the parallel transformation from the tangent space at p onto the tangent space at q along γ .

Lemma 2.18 *Let $q \in \text{supp } \varphi_i$. Then, for any $\tilde{u} \in \mathbb{S}_q^{n-1} := \{v \in T_q M \mid \|v\| = 1\}$,*

$$(2.17) \quad \|(dF_\varepsilon^{(p_i)})_q(\tilde{u})\| \leq \text{Lip}(F) \left(1 + \sup_{y \in \mathbb{B}_\varepsilon(o_{p_i})} \|Y_y^{(\tilde{u})}(1) - \tau_{q_i(y)}^q(\tilde{u})\|\right),$$

holds, where $q_i(y) := \exp_{p_i}(\exp_{p_i}^{-1} q - y)$, and $Y_y^{(\tilde{u})}(t) := \frac{\partial}{\partial s} \exp_{p_i} t (\exp_{p_i}^{-1}(\exp_q s \tilde{u}) - y) \big|_{s=0}$ is a Jacobi field along the geodesic $\exp_{p_i} t (\exp_{p_i}^{-1} q - y)$ for each y .

Proof. From (2.15), it is easy to obtain

$$(2.18) \quad (dF_\varepsilon^{(p_i)})_q(\tilde{u}) = \int_{\mathbb{R}^n} \rho_\varepsilon(y) dF_{q_i(y)}(Y_y^{(\tilde{u})}(1)) dy$$

and $dF_{q_i(y)}(Y_y^{(\tilde{u})}(1)) = dF_{q_i(y)}(Y_y^{(\tilde{u})}(1) - \tau_{q_i(y)}^q(\tilde{u})) + dF_{q_i(y)}(\tau_{q_i(y)}^q(\tilde{u}))$. Hence, by the triangle inequality, we get (2.17). \square

Lemma 2.19 *For any $\eta > 0$, there exists a number $\varepsilon_i(\eta) > 0$ such that*

$$\sup\{\|Y_y^{(\tilde{u})}(1) - \tau_{q_i(y)}^q(\tilde{u})\| \mid q \in \text{supp } \varphi_i, \tilde{u} \in \mathbb{S}_q^{n-1}, y \in \mathbb{B}_{\varepsilon_i(\eta)}(o_{p_i})\} < \eta.$$

Proof. In the similar way of the proof of Lemma 2.5, we get the inequality. \square

Lemma 2.20 *For any $\eta > 0$, there exists a number $\varepsilon(\eta) > 0$ such that*

$$(2.19) \quad \|dF_\varepsilon(\tilde{u})\| \leq (1 + \eta) \text{Lip}(F)$$

holds for all $\varepsilon \in (0, \varepsilon(\eta))$ and all unit tangent vectors \tilde{u} on M .

Proof. Take any $\tilde{u} \in \mathbb{S}_q^{n-1}$, and fix it. Since $\sum_{i=1}^\ell \varphi_i = 1$ on M , we get $\sum_{i=1}^\ell (d\varphi_i)_q(\tilde{u}) = 0$. Then, we have

$$(2.20) \quad (dF_\varepsilon)_q(\tilde{u}) = \sum_{i=1}^\ell \varphi_i(q) (dF_\varepsilon^{(p_i)})_q(\tilde{u}) + \sum_{i=1}^\ell (d\varphi_i)_q(\tilde{u}) (F_\varepsilon^{(p_i)}(q) - F(q)).$$

By applying the triangle inequality to the equation above, we have

$$\|(dF_\varepsilon)_q(\tilde{u})\| \leq \sum_{i=1}^\ell \varphi_i(q) \|(dF_\varepsilon^{(p_i)})_q(\tilde{u})\| + \sum_{i=1}^\ell |(d\varphi_i)_q(\tilde{u})| \cdot \|F_\varepsilon^{(p_i)}(q) - F(q)\|.$$

By Lemmas 2.16, 2.18 and 2.19, we get (2.19) for all sufficiently small $\varepsilon > 0$. \square

From now on, we assume that $n = \dim M \leq \dim N = k$.

Lemma 2.21 *For each non-singular point $p \in M$ of F , there exist positive numbers $r(p)$ and $\delta(p)$ such that, for any $u \in \mathbb{S}_p^{n-1}$, there exists a local unit vector field V on a neighborhood of $F(p)$ satisfying*

$$(2.21) \quad \langle dF_q(\tau_q^p(u)), V_{F(q)} \rangle \geq \delta(p)$$

for almost all $q \in B_{2r(p)}(p)$.

Proof. Choose convex balls $B_{r_1}(p)$ and $B_{r_2}(F(p))$ respectively so as to satisfy $F(B_{r_1}(p)) \subset B_{r_2}(F(p))$. Since the point p is a non-singular point of F , it follows from Example 1.8 that there exist positive numbers $r(p) \in (0, r_1/2)$ and $\delta(p)$ satisfying the following property: For any $u \in \mathbb{S}_p^{n-1}$ at the point p , there exists a unit vector v at $F(p)$ such that

$$\langle \tau_{F(p)}^{F(q)} \circ dF_q \circ \tau_q^p(u), v \rangle \geq \delta(p)$$

for almost all $q \in B_{2r(p)}(p)$. Hence, we get $\langle dF_q(\tau_q^p(u)), V_{F(q)} \rangle \geq \delta(p)$ for almost all $q \in B_{2r(p)}(p)$, where $V_{F(q)} := \tau_{F(q)}^{F(p)}(v)$. \square

Henceforth, we fix any non-singular point $p \in M$ of F and any $\tilde{u} \in \mathbb{S}_q^{n-1}$ at any point $q \in B_{r(p)}(p)$. Here, we also fix an integer $i \in \{1, 2, \dots, \ell\}$ satisfying $q \in \text{supp } \varphi_i \subset B_{r_i}(p_i)$.

Lemma 2.22 *There exists a unit vector field V on a neighborhood of $F(p)$ such that*

$$(2.22) \quad \begin{aligned} & \langle (dF_\varepsilon^{(p_i)})_q(\tilde{u}), V_{F(q)} \rangle \\ & \geq -\text{Lip}(F) \left(\sup_{y \in \mathbb{B}_\varepsilon(o_{p_i})} \|Y_y^{(\tilde{u})}(1) - U_{q_i(y)}\| + \sup_{y \in \mathbb{B}_\varepsilon(o_{p_i})} \|V_{F(q)} - V_{F(q_i(y))}\| \right) + \delta(p) \end{aligned}$$

on $\varepsilon \in (0, \varepsilon_i(p))$. Here, $\varepsilon_i(p) := \min \{r_i, r(p)/\text{Lip}(\exp_{p_i}|_{\mathbb{B}_{2r_i}(o_{p_i})})\}$ and $U_{q_i(y)} := \tau_{q_i(y)}^p \circ \tau_p^q(\tilde{u})$.

Proof. It follows from Lemma 2.21 that for the unit tangent vector $u := \tau_p^q(\tilde{u})$, there exists a unit vector field V on a neighborhood of $F(p)$ satisfying (2.21). By the triangle inequality, $\|\exp_{p_i}^{-1} q - y\| < r_i + \|y\| < 2r_i$ for all $y \in \mathbb{B}_{r_i}(o_{p_i})$. Thus, we get

$$d_M(q, q_i(y)) \leq \|y\| \cdot \text{Lip}(\exp_{p_i}|_{\mathbb{B}_{2r_i}(o_{p_i})})$$

for all $y \in \mathbb{B}_{r_i}(o_{p_i})$, where d_M denotes the distance function of M . Then, from the triangle inequality, we obtain

$$d_M(p, q_i(y)) \leq d_M(p, q) + d_M(q, q_i(y)) < 2r(p)$$

for all $y \in \mathbb{B}_{\varepsilon_i(p)}(o_{p_i})$. Thus, by Lemma 2.21,

$$(2.23) \quad \langle dF_{q_i(y)}(U_{q_i(y)}), V_{F(q_i(y))} \rangle \geq \delta(p)$$

for almost all $y \in \mathbb{B}_{\varepsilon_i(p)}(o_{p_i})$, where $U_{q_i(y)} := \tau_{q_i(y)}^p(u)$. It is clear to see that for almost all $y \in \mathbb{B}_{\varepsilon}(o_{p_i})$,

$$(2.24) \quad \begin{aligned} & \langle dF_{q_i(y)}(U_{q_i(y)}), V_{F(q)} \rangle \\ &= \langle dF_{q_i(y)}(U_{q_i(y)}), V_{F(q)} - V_{F(q_i(y))} \rangle + \langle dF_{q_i(y)}(U_{q_i(y)}), V_{F(q_i(y))} \rangle \\ &\geq -\text{Lip}(F) \sup_{y \in \mathbb{B}_{\varepsilon}(o_{p_i})} \|V_{F(q)} - V_{F(q_i(y))}\| + \langle dF_{q_i(y)}(U_{q_i(y)}), V_{F(q_i(y))} \rangle. \end{aligned}$$

Since $dF_{q_i(y)}(Y_y^{\tilde{u}}(1)) = dF_{q_i(y)}(Y_y^{\tilde{u}}(1) - U_{q_i(y)}) + dF_{q_i(y)}(U_{q_i(y)})$, we therefore get (2.22) from (2.18), (2.23) and (2.24). \square

Lemma 2.23 *For any $\eta > 0$, there exists a number $\varepsilon_i(p, \eta) > 0$ such that*

$$\sup\{\|Y_y^{\tilde{u}}(1) - U_{q_i(y)}\| \mid q \in \text{supp } \varphi_i \cap \overline{B_{r(p)}(p)}, \tilde{u} \in \mathbb{S}_q^{n-1}, y \in \mathbb{B}_{\varepsilon_i(p, \eta)}(o_{p_i})\} < \eta$$

and

$$\sup\{\|V_{F(q)} - V_{F(q_i(y))}\| \mid q \in \text{supp } \varphi_i \cap \overline{B_{r(p)}(p)}, v \in \mathbb{S}_{F(p)}^{k-1}, y \in \mathbb{B}_{\varepsilon_i(p, \eta)}(o_{p_i})\} < \eta,$$

where $\overline{B_{r(p)}(p)} := \{q \in M \mid d(p, q) \leq r(p)\}$ and $V_{F(q)} := \tau_{F(q)}^{F(p)}(v)$.

Proof. Imitate the proof of Lemma 2.19. \square

Lemma 2.24 *There exists $V_{F(q)} \in \mathbb{S}_{F(q)}^{k-1}$ such that*

$$\langle (dF_{\varepsilon}^{(p_i)})_q(\tilde{u}), V_{F(q)} \rangle \geq \frac{2\delta(p)}{3}$$

for all $\varepsilon \in (0, \varepsilon(p))$ and all p_i with $q \in \text{supp } \varphi_i$. Here we set

$$\varepsilon(p) := \min\{\varepsilon_i(p), \varepsilon_i(p, \eta_0) \mid \text{supp } \varphi_i \cap B_{r(p)}(p) \neq \emptyset\},$$

where $\eta_0 := \delta(p)/6 \text{Lip}(F)$.

Proof. The inequality is immediate from Lemmas 2.22 and 2.23. \square

Lemma 2.25 *There exist $V_{F(q)} \in \mathbb{S}_{F(q)}^{k-1}$ and a number $\varepsilon_0(p) > 0$ such that*

$$(2.25) \quad \langle (dF_\varepsilon)_q(\tilde{u}), V_{F(q)} \rangle \geq \frac{1}{3}\delta(p)$$

for all $\varepsilon \in (0, \varepsilon_0(p))$.

Proof. By Lemma 2.24, for any $\varepsilon \in (0, \varepsilon(p))$,

$$(2.26) \quad \sum_{i=1}^{\ell} \varphi_i(q) \langle dF_\varepsilon^{(p_i)}(\tilde{u}), V_{F(q)} \rangle \geq \frac{2}{3}\delta(p).$$

From Lemma 2.16, we may choose a number $\varepsilon_0(p) \in (0, \varepsilon(p))$ satisfying

$$(2.27) \quad \left| \sum_{i=1}^{\ell} d\varphi_i(\tilde{u}) \langle F_\varepsilon^{(p_i)}(q) - F(q), V_{F(q)} \rangle \right| < \frac{1}{3}\delta(p)$$

for all $\varepsilon \in (0, \varepsilon_0(p))$. Combining (2.20), (2.26) and (2.27), we get (2.25). \square

(*Proof of Theorem 1.3*) Let $F : M \rightarrow N$ be a Lipschitz map from a compact Riemannian manifold M into a Riemannian manifold N , where $\dim M \leq \dim N$. Assume that F has no singular points on M . We embed N into \mathbb{R}^m , where $m \geq \dim N + 1$, and introduce the induced metric from the outer space to N . Note that $F : M \rightarrow \mathbb{R}^m$. Since N is a smooth submanifold in \mathbb{R}^m , for any $x \in N$ there exists an open neighborhood U_x of x in \mathbb{R}^m such that each $y \in U_x$ admits a unique $z \in N$ with $\|y - z\| = \inf_{w \in N} \|y - w\|$. Considering an open set $\mathcal{U}_N := \cup_{x \in N} U_x$ in \mathbb{R}^m , we have a smooth locally distance projection $\pi_N : \mathcal{U}_N \rightarrow N$. Note here that $N \subset \mathcal{U}_N$. Since $F(M)$ is compact, it follows from Lemma 2.17 that for all sufficiently small $\varepsilon > 0$, the image of the map F_ε defined by (2.16) is a subset of \mathcal{U}_N . Then, for any sufficiently small $\varepsilon > 0$, we can define a smooth map $\pi_N \circ F_\varepsilon$ from M into N . By its definition, $(d\pi_N)_x$ is an orthogonal projection to $T_x N$ for each $x \in N$. Therefore, for any sufficiently small $\varepsilon > 0$, the map $\pi_N \circ F_\varepsilon : M \rightarrow N$ is an immersion from Lemma 2.25. Combining Lemmas 2.17, 2.20 and the argument above, we get the assertion. \square

2.3 Proof of Corollary 1.10

We need one more lemma in order to prove Corollary 1.10. In what follows, d_M denotes the distance function of a given Riemannian manifold M .

Lemma 2.26 *Let M be a compact (connected) Riemannian manifold, and let $F : M \rightarrow M$ be C^∞ and a local diffeomorphism. If $\max_{x \in M} d_M(F(x), x)$ is sufficiently small, then F is injective. In particular, F is a diffeomorphism.*

Proof. Since $F(M)$ is open and closed in M , F is surjective. Then, we can consider F to be a covering map. Since M is compact, there exists a number a such that $0 < a < i(M)$, where $i(M)$ denotes the injectivity radius of M . Since $\max_{x \in M} d_M(F(x), x) \ll 1$, we may assume that $d_M(F(x), x) < i(M)$ for all $x \in M$. Thus, for each $x \in M$, there exists the unique minimal geodesic segment $\gamma : [0, 1] \rightarrow M$ joining $x = \gamma(0)$ to $F(x) = \gamma(1)$. Define a map $H : M \times [0, 1] \rightarrow M$ by $H(x, t) := \sigma_x(t)$, where $\sigma_x : [0, 1] \rightarrow M$ denotes the unique minimal geodesic segment emanating from $x = \sigma_x(0)$ to $F(x) = \sigma_x(1)$. H is continuous, for F is continuous and σ_x is unique. By the definition of H , $H(x, 0) = x = \text{id}_M(x)$ and $H(x, 1) = F(x)$ for all $x \in M$, i.e., F is homotopic to id_M . Thus, the homomorphism $F_\# : \pi_1(M, x) \rightarrow \pi_1(M, F(x))$ induced by F is equal to an isomorphism induced from the path σ_x , and hence $\pi_1(M, x)$ and $\pi_1(M, F(x))$ are isomorphic. Since $\pi_1(M, F(x))/F_\#(\pi_1(M, x))$ is trivial, it follows from a well known lemma (cf. Corollary 3 in [33, Chapter 3]) that $F^{-1}(x)$ is one point, i.e., F is injective. \square

(*Proof of Corollary 1.10*) Let F be a bi-Lipschitz homeomorphism from a compact Riemannian manifold M onto a Riemannian manifold N . Assume that F and F^{-1} have no singular points on M and N , respectively. By Theorem 1.3, for any $\eta > 0$, there exist two smooth immersions f_η from M into N and g_η from N into M such that

$$\max_{p \in M} d_N(f_\eta(p), F(p)) < \eta, \quad \text{Lip}(f_\eta) \leq \text{Lip}(F)(1 + \eta),$$

and that

$$\max_{q \in N} d_M(g_\eta(q), F^{-1}(q)) < \eta, \quad \text{Lip}(g_\eta) \leq \text{Lip}(F^{-1})(1 + \eta),$$

respectively. Hence, $\text{Lip}(g_\eta \circ f_\eta) \leq \text{Lip}(g_\eta) \text{Lip}(f_\eta) \leq \text{Lip}(F) \text{Lip}(F^{-1})(1 + \eta)^2$. Moreover, by the triangle inequality, we get

$$d_M(g_\eta \circ f_\eta(p), p) \leq d_M(g_\eta(f_\eta(p)), F^{-1}(f_\eta(p))) + d_M(F^{-1}(f_\eta(p)), F^{-1}(F(p))).$$

Since $d_M(g_\eta(f_\eta(p)), F^{-1}(f_\eta(p))) < \eta$ and $d_M(F^{-1}(f_\eta(p)), F^{-1}(F(p))) < \eta \text{Lip}(F^{-1})$ for all $p \in M$, we obtain $\max_{p \in M} d_M(g_\eta \circ f_\eta(p), p) < \eta(1 + \text{Lip}(F^{-1}))$. Therefore, by Lemma 2.26, for all sufficiently small $\eta > 0$, $g_\eta \circ f_\eta$ is a diffeomorphism on M . This implies that f_η and g_η are injective for all sufficiently small η , and hence M and N are diffeomorphic. \square

3 Proofs of the differentiable exotic sphere theorems

We first give two preliminaries to three cases (1.6), (1.7), and (1.10), respectively.

3.1 Preliminaries to the proofs in assuming (1.6) and (1.10)

Throughout this subsection, let $\sigma : S^{n-1}(1) := \{v \in \mathbb{R}^n \mid \|v\| = 1\} \rightarrow S^{n-1}(1)$ denote a smooth map satisfying

$$(3.1) \quad \int_0^\pi e^{-t} \|\ddot{c}(t) + c(t)\| dt \leq e^{-\pi} \alpha$$

for some $\alpha > 0$ and any unit speed geodesic $\gamma : [0, \pi] \rightarrow S^{n-1}(1)$. Here,

$$c := \sigma \circ \gamma, \quad \dot{c} := \frac{dc}{dt}, \quad \text{and} \quad \ddot{c} := \frac{d^2c}{dt^2}.$$

Our aim in this subsection is to prove the following theorem:

Theorem 3.1 *If $\alpha > 0$ is sufficiently small, for example $\alpha = 1 - 1/\sqrt{2}$, then the origin $o \in \mathbb{R}^n$ is non-singular of a map $\tilde{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by*

$$\tilde{F}(v) := \begin{cases} \|v\| \sigma\left(\frac{v}{\|v\|}\right) & \text{on } \mathbb{R}^n \setminus \{o\}, \\ o & \text{when } v = o. \end{cases}$$

We need six lemmas to prove Theorem 3.1. So, we first prove the lemmas before proving the theorem: Take any unit speed geodesic $\gamma : [0, \pi] \rightarrow S^{n-1}(1)$, and fix it. Set

$$\bar{c}(t) := c(0) \cos t + \dot{c}(0) \sin t$$

on $[0, \pi]$.

Lemma 3.2 *For any $t \in [0, \pi]$,*

$$\sqrt{\|c(t) - \bar{c}(t)\|^2 + \|\dot{c}(t) - \dot{\bar{c}}(t)\|^2} \leq e^\pi \int_0^\pi e^{-\theta} \|\ddot{c}(\theta) + c(\theta)\| d\theta$$

holds. In particular, we have

$$(3.2) \quad \|c(t) - \bar{c}(t)\| \leq \alpha$$

for all $t \in [0, \pi]$.

Proof. Let $f(t) := \sqrt{\|c(t) - \bar{c}(t)\|^2 + \|\dot{c}(t) - \dot{\bar{c}}(t)\|^2}$, and let $X(t) := \begin{pmatrix} c(t) \\ \dot{c}(t) \end{pmatrix}$, $\bar{X}(t) := \begin{pmatrix} \bar{c}(t) \\ \dot{\bar{c}}(t) \end{pmatrix} \in \mathbb{R}^{2n}$. Then, $f(t) = \|X(t) - \bar{X}(t)\|$. Note that $f(0) = 0$ because $\bar{c}(0) = c(0)$ and $\dot{\bar{c}}(0) = \dot{c}(0)$. Choose an open interval $(a, b) \subset [0, \pi]$ such that $f(t) > 0$ on (a, b) and $f(a) = 0$. In this situation, we observe

$$(3.3) \quad \begin{aligned} f'(t) &= \frac{1}{\|X(t) - \bar{X}(t)\|} \left\langle X(t) - \bar{X}(t), X'(t) - \bar{X}'(t) \right\rangle \\ &\leq \|X'(t) - \bar{X}'(t)\| = \sqrt{\|\dot{c}(t) - \dot{\bar{c}}(t)\|^2 + \|\ddot{c}(t) - \ddot{\bar{c}}(t)\|^2} \end{aligned}$$

on (a, b) . Since $\ddot{\bar{c}}(t) = -\bar{c}(t)$, we have

$$(3.4) \quad \begin{aligned} \|\ddot{c}(t) - \ddot{\bar{c}}(t)\| &= \|(\ddot{c}(t) + c(t)) - (\ddot{\bar{c}}(t) + c(t))\| \\ &\leq \|\ddot{c}(t) + c(t)\| + \|\ddot{\bar{c}}(t) + c(t)\| = \|\ddot{c}(t) + c(t)\| + \|c(t) - \bar{c}(t)\|. \end{aligned}$$

Thus, by (3.4),

$$\begin{aligned}
(3.5) \quad & \|\dot{c}(t) - \bar{\dot{c}}(t)\|^2 + \|\ddot{c}(t) - \bar{\ddot{c}}(t)\|^2 \\
& \leq \|\dot{c}(t) - \bar{\dot{c}}(t)\|^2 + \|\ddot{c}(t) + c(t)\|^2 + 2\|\ddot{c}(t) + c(t)\| \cdot \|c(t) - \bar{c}(t)\| + \|c(t) - \bar{c}(t)\|^2 \\
& \leq f^2(t) + 2\|\ddot{c}(t) + c(t)\|f(t) + \|\ddot{c}(t) + c(t)\|^2 = (f(t) + \|\ddot{c}(t) + c(t)\|)^2.
\end{aligned}$$

Hence, by (3.3) and (3.5), we get

$$(3.6) \quad f'(t) \leq f(t) + \|\ddot{c}(t) + c(t)\|$$

on (a, b) . Since $e^{-t}(f'(t) - f(t)) \leq e^{-t}\|\ddot{c}(t) + c(t)\|$ from (3.6), and since $f(a) = 0$,

$$\int_a^b e^{-t}\|\ddot{c}(t) + c(t)\|dt \geq \int_a^b e^{-t}(f'(t) - f(t))dt = \int_a^b (e^{-t}f(t))'dt = e^{-b}f(b),$$

and hence $f(b) \leq e^b \int_a^b e^{-\theta}\|\ddot{c}(\theta) + c(\theta)\|d\theta \leq e^b \int_0^b e^{-\theta}\|\ddot{c}(\theta) + c(\theta)\|d\theta$. Since a function $t \rightarrow e^t \int_0^t e^{-\theta}\|\ddot{c}(\theta) + c(\theta)\|d\theta$ is increasing on $[0, \pi]$,

$$(3.7) \quad f(b) \leq e^b \int_0^b e^{-\theta}\|\ddot{c}(\theta) + c(\theta)\|d\theta \leq e^\pi \int_0^\pi e^{-\theta}\|\ddot{c}(\theta) + c(\theta)\|d\theta.$$

If $f(t) = 0$ for some $t \in [0, \pi]$, then (3.7) still holds for such a t . Therefore,

$$f(t) \leq e^\pi \int_0^\pi e^{-\theta}\|\ddot{c}(\theta) + c(\theta)\|d\theta.$$

holds on $[0, \pi]$. □

Lemma 3.3 $\langle c(0), c(\theta) \rangle \cos \theta \geq \cos^2 \theta - \alpha |\cos \theta|$ for all $\theta \in [0, \pi]$.

Proof. Since $\|c(0)\| = 1$ and $c(0) \perp \dot{c}(0)$,

$$\begin{aligned}
(3.8) \quad \langle c(0), c(\theta) \rangle &= \langle c(0), c(\theta) - \bar{c}(\theta) \rangle + \langle c(0), \bar{c}(\theta) \rangle \\
&= \langle c(0), c(\theta) - \bar{c}(\theta) \rangle + \langle c(0), c(0) \cos \theta + \dot{c}(0) \sin \theta \rangle \\
&= \langle c(0), c(\theta) - \bar{c}(\theta) \rangle + \cos \theta.
\end{aligned}$$

By (3.2) and (3.8), we have

$$\langle c(0), c(\theta) \rangle \cos \theta \geq \cos^2 \theta - \|c(\theta) - \bar{c}(\theta)\| \cdot |\cos \theta| \geq \cos^2 \theta - \alpha |\cos \theta|.$$

□

Lemma 3.4 $\langle \dot{c}(0), c(\theta) \rangle \sin \theta \geq \|\dot{c}(0)\|^2 \sin^2 \theta - \alpha \|\dot{c}(0)\| \sin \theta$ for all $\theta \in [0, \pi]$.

Proof. Since $c(0) \perp \dot{c}(0)$,

$$(3.9) \quad \langle \dot{c}(0), c(\theta) \rangle = \langle \dot{c}(0), c(\theta) - \bar{c}(\theta) \rangle + \langle \dot{c}(0), \bar{c}(\theta) \rangle = \langle \dot{c}(0), c(\theta) - \bar{c}(\theta) \rangle + \|\dot{c}(0)\|^2 \sin \theta.$$

By (3.2) and (3.9), we have

$$\begin{aligned} \langle \dot{c}(0), c(\theta) \rangle \sin \theta &\geq \|\dot{c}(0)\|^2 \sin^2 \theta - \|\dot{c}(0)\| \cdot \|c(\theta) - \bar{c}(\theta)\| \sin \theta \\ &\geq \|\dot{c}(0)\|^2 \sin^2 \theta - \alpha \|\dot{c}(0)\| \sin \theta. \end{aligned}$$

□

Lemma 3.5 $\|\dot{c}(0)\| - 1 \leq \alpha$.

Proof. Since $\bar{c}(\pi/2) = \dot{c}(0)$, $\dot{c}(0) = \bar{c}(\pi/2) - c(\pi/2) + c(\pi/2)$. Since $\|c(\pi/2)\| = 1$, by the triangle inequality and (3.2), $\|\dot{c}(0)\| \leq \|\bar{c}(\pi/2) - c(\pi/2)\| + \|c(\pi/2)\| \leq \alpha + 1$, and $\|\dot{c}(0)\| \geq \|c(\pi/2)\| - \|\bar{c}(\pi/2) - c(\pi/2)\| \geq 1 - \alpha$. Hence, we get the assertion. □

Lemma 3.6 For $\alpha = 1 - 1/\sqrt{2}$, we have

$$\langle \bar{c}(\theta), c(\theta) \rangle > 0$$

on $[0, \pi]$.

Proof. First we consider the case where $\|\dot{c}(0)\| \geq 1$. Then, by Lemmas 3.3, 3.4, and 3.5, we have

$$\begin{aligned} (3.10) \quad \langle \bar{c}(\theta), c(\theta) \rangle &= \langle c(0), c(\theta) \rangle \cos \theta + \langle \dot{c}(0), c(\theta) \rangle \sin \theta \\ &\geq \cos^2 \theta - \alpha |\cos \theta| + \|\dot{c}(0)\|^2 \sin^2 \theta - \alpha \|\dot{c}(0)\| \sin \theta \\ &\geq \cos^2 \theta - \alpha |\cos \theta| + \sin^2 \theta - \alpha(\alpha + 1) \sin \theta \\ &= 1 - \alpha |\cos \theta| - \alpha(\alpha + 1) \sin \theta. \end{aligned}$$

Set $t = \sin \theta$. Note that $0 \leq t \leq 1$. Since $|\cos \theta| = \sqrt{1 - t^2} \leq 1 - t^2/2$, we have, by (3.10),

$$\begin{aligned} (3.11) \quad \langle \bar{c}(\theta), c(\theta) \rangle &\geq 1 - \alpha |\cos \theta| - \alpha(\alpha + 1) \sin \theta \\ &\geq 1 + \alpha \left(\frac{t^2}{2} - 1 \right) - \alpha(\alpha + 1)t \\ &= \frac{\alpha}{2} t^2 - \alpha(\alpha + 1)t + 1 - \alpha \\ (3.12) \quad &= \frac{\alpha}{2} \{t - (\alpha + 1)\}^2 - \frac{\alpha}{2}(\alpha + 1)^2 + 1 - \alpha. \end{aligned}$$

Consider a function $\varphi(t) := (\alpha/2)t^2 - \alpha(\alpha + 1)t + 1 - \alpha$ on $[0, 1]$, which is a parabola. From (3.12), the axis of φ is $t = \alpha + 1$. Since $\alpha = 1 - 1/\sqrt{2}$, we see that $\alpha/2 > 0$ and $\alpha + 1 > 1$, which imply that φ is decreasing on $[0, 1]$. Moreover, since $\alpha = 1 - 1/\sqrt{2}$, we see that

$$(3.13) \quad \varphi(1) = -\left(\alpha + \frac{3}{4}\right)^2 + \frac{25}{16} > 0.$$

Hence, by (3.11), (3.12), and (3.13),

$$\langle \bar{c}(\theta), c(\theta) \rangle \geq \varphi(1) > 0$$

holds on $[0, \pi]$.

Finally we consider the case where $\|\dot{c}(0)\| < 1$. Then, since $\alpha = 1 - 1/\sqrt{2}$, we have, by Lemmas 3.3, 3.4, and 3.5,

$$\begin{aligned} \langle \bar{c}(\theta), c(\theta) \rangle &= \langle c(0), c(\theta) \rangle \cos \theta + \langle \dot{c}(0), c(\theta) \rangle \sin \theta \\ &\geq \cos^2 \theta - \alpha |\cos \theta| + \|\dot{c}(0)\|^2 \sin^2 \theta - \alpha \|\dot{c}(0)\| \sin \theta \\ &> \cos^2 \theta + \sin^2 \theta - \alpha |\cos \theta| + (\|\dot{c}(0)\|^2 - 1) \sin^2 \theta - \alpha \sin \theta \\ &= 1 - \alpha (|\cos \theta| + \sin \theta) + (\|\dot{c}(0)\| + 1)(\|\dot{c}(0)\| - 1) \sin^2 \theta \\ &\geq 1 - \alpha (|\cos \theta| + \sin \theta) - \alpha (\|\dot{c}(0)\| + 1) \sin^2 \theta \\ &> 1 - \sqrt{2}\alpha - 2\alpha = 0 \end{aligned}$$

for all $\theta \in [0, \pi]$. □

Lemma 3.7 *Let $\tilde{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a map defined by*

$$(3.14) \quad \tilde{F}(v) := \begin{cases} \|v\| \sigma\left(\frac{v}{\|v\|}\right) & \text{on } \mathbb{R}^n \setminus \{o\}, \\ o & \text{when } v = o. \end{cases}$$

Then, the generalized differential of \tilde{F} at o is given by

$$\partial \tilde{F}(o) = \text{Conv}(\{A_v \mid v \in S^{n-1}(1)\}),$$

where A_v is a linear map defined by $A_v(\lambda v) = \lambda \sigma(v)$ for all $\lambda \in \mathbb{R}$ and $A_v(w) = d\sigma_v(w)$ for all $w \in \mathbb{R}^n$ with $w \perp v$.

Proof. Let $v \in S^{n-1}(1)$, $\ell > 0$, and $\lambda \in \mathbb{R}$ constant number. By (3.14),

$$(3.15) \quad d\tilde{F}_{\ell v}(\lambda v) = \left. \frac{d}{dt} \tilde{F}(\ell v + t\lambda v) \right|_{t=0} = \left. \frac{d}{dt} (\ell + \lambda t) \sigma(v) \right|_{t=0} = \lambda \sigma(v).$$

For any $w \in \mathbb{R}^n$ with $w \perp v$, by (3.14),

$$\begin{aligned} (3.16) \quad d\tilde{F}_{\ell v}(w) &= \left. \frac{d}{dt} \tilde{F}(\ell v + tw) \right|_{t=0} = \left. \frac{d}{dt} \tilde{F} \left(\sqrt{\ell^2 + \|w\|^2 t^2} \cdot \frac{\ell v + tw}{\sqrt{\ell^2 + \|w\|^2 t^2}} \right) \right|_{t=0} \\ &= \left. \frac{d}{dt} \sqrt{\ell^2 + \|w\|^2 t^2} \cdot \sigma \left(\frac{\ell v + tw}{\sqrt{\ell^2 + \|w\|^2 t^2}} \right) \right|_{t=0} \\ &= \ell \cdot \left. \frac{d}{dt} \sigma \left(\frac{\ell v + tw}{\sqrt{\ell^2 + \|w\|^2 t^2}} \right) \right|_{t=0} \\ &= \ell \cdot d\sigma_v \left(\frac{w}{\ell} \right) = d\sigma_v(w) \end{aligned}$$

Hence, by (3.15) and (3.16), we obtain $\partial \tilde{F}(o) = \text{Conv}(\{A_v \mid v \in S^{n-1}(1)\})$. □

Remark 3.8 In the proof of Lemma 3.7, we do not need the assumption that σ satisfies (3.1) for some α .

Now, we are going to give the proof of Theorem 3.1:

(Proof of Theorem 3.1) Let $\alpha > 0$ be sufficiently small, and let $\tilde{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the map defined by (3.14). Let $u, v \in S^{n-1}(1)$ be any vectors. Then, we can take a unit speed geodesic $\gamma : [0, \pi] \rightarrow S^{n-1}(1)$ emanating from $v = \gamma(0)$ satisfying $u = \gamma(t_0)$ for some $t_0 \in [0, \pi]$. Setting $w := \dot{\gamma}(0)$, we have $\gamma(t) = v \cos t + w \sin t$ on $[0, \pi]$. Take any $\ell > 0$, and fix it. Since $c(0) = \sigma(\gamma(0)) = \sigma(v)$ and $\dot{c}(0) = d\sigma_{\gamma(0)}(\dot{\gamma}(0)) = d\sigma_v(w)$, we see, by (3.15) and (3.16),

$$\begin{aligned} d\tilde{F}_{\ell v}(u) &= d\tilde{F}_{\ell v}(\gamma(t_0)) = d\tilde{F}_{\ell v}(v) \cos t_0 + d\tilde{F}_{\ell v}(w) \sin t_0 \\ &= \sigma(v) \cos t_0 + d\sigma_v(w) \sin t_0 \\ &= c(0) \cos t_0 + \dot{c}(0) \sin t_0 = \bar{c}(t_0). \end{aligned}$$

Since $\sigma(u) = \sigma(\gamma(t_0)) = c(t_0)$,

$$(3.17) \quad \langle d\tilde{F}_{\ell v}(u), \sigma(u) \rangle = \langle d\tilde{F}_{\ell v}(u), c(t_0) \rangle = \langle \bar{c}(t_0), c(t_0) \rangle.$$

Thus, if $\alpha = 1 - 1/\sqrt{2}$, then, by Lemma 3.6, $\langle d\tilde{F}_{\ell v}(u), \sigma(u) \rangle > 0$ holds. This implies

$$(3.18) \quad \langle A_v(u), \sigma(u) \rangle > 0$$

for all $u, v \in S^{n-1}(1)$. Therefore, by Lemma 3.7 and (3.18), o is non-singular of \tilde{F} . Indeed, suppose that o is singular of \tilde{F} . Set $G := \{A_v \mid v \in S^{n-1}(1)\}$. Since

$$\text{Conv}(G) = \left\{ \sum_{i=1}^{\ell} \lambda_i A_{z_i} \mid A_{z_i} \in G, \sum_{i=1}^{\ell} \lambda_i = 1, \lambda_i \geq 0 \ (i = 1, 2, \dots, \ell) \right\},$$

where $\ell < \infty$ (by Carathéodory's theorem), there exists $\sum_{i=1}^{\ell} \lambda_i A_{z_i} \in \partial \tilde{F}(o)$ such that $\text{rank}(\sum_{i=1}^{\ell} \lambda_i A_{z_i}) < n$. Thus, we may find a vector $v_0 \in S^{n-1}(1)$ such that $\sum_{i=1}^{\ell} \lambda_i A_{z_i}(v_0) = o$. By (3.18),

$$0 = \langle o, \sigma(v_0) \rangle = \left\langle \sum_{i=1}^{\ell} \lambda_i A_{z_i}(v_0), \sigma(v_0) \right\rangle = \sum_{i=1}^{\ell} \lambda_i \langle A_{z_i}(v_0), \sigma(v_0) \rangle > 0,$$

which is a contradiction. Hence, o is non-singular of \tilde{F} . \square

From (3.17) in the proof of Theorem 3.1, we have the following lemma, which is applied to the proof of the differentiable exotic sphere theorem II.

Lemma 3.9 *If σ satisfies*

$$\angle(\bar{c}(t), c(t)) < \frac{\pi}{2}$$

for all geodesic segments $\gamma([0, \pi]) \subset S^{n-1}(1)$ with $\|\dot{\gamma}\| \equiv 1$, then $o \in \mathbb{R}^n$ is non-singular of \tilde{F} defined by (3.14). Here, we do not assume that σ satisfies (3.1) for some α .

We need a corollary below of Theorem 3.1 to prove the differentiable exotic sphere theorem I in the case where (1.6). We prepare the following lemma for proving the corollary.

Lemma 3.10 *A smooth curve $c : (a, b) \longrightarrow S^{n-1}(1) \subset \mathbb{R}^n$ satisfies*

$$\|\ddot{c}(t) + c(t)\|^2 = \|\ddot{c}(t)\|^2 - 2\|\dot{c}(t)\|^2 + 1.$$

Proof. It is clear that $\|\ddot{c}(t) + c(t)\|^2 = \|\ddot{c}\|^2 + 2\langle \ddot{c}(t), c(t) \rangle + 1$, where note that $\|c(t)\| \equiv 1$ on (a, b) . Since $1 = \|c(t)\|^2 = \langle c(t), c(t) \rangle$ for all $t \in (a, b)$, $\langle \dot{c}(t), c(t) \rangle \equiv 0$ holds. We thus have $\langle \ddot{c}(t), c(t) \rangle = -\langle \dot{c}(t), \dot{c}(t) \rangle = -\|\dot{c}(t)\|^2$. Hence, we get

$$\|\ddot{c}(t) + c(t)\|^2 = \|\ddot{c}\|^2 + 2\langle \ddot{c}(t), c(t) \rangle + 1 = \|\ddot{c}(t)\|^2 - 2\|\dot{c}(t)\|^2 + 1.$$

□

Corollary 3.11 *Let $\sigma : S^{n-1}(1) \longrightarrow S^{n-1}(1)$ be a smooth map, $c := \sigma \circ \gamma$ a curve in $S^{n-1}(1)$, where $\gamma : [0, \pi] \longrightarrow S^{n-1}(1)$ denotes a geodesic segment, and $\tilde{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the map defined by (3.14). If $\text{Lip}^b(\sigma)$ and $\|\ddot{c}\|$ satisfy (1.6) for all geodesic segments $\gamma([0, \pi]) \subset S^{n-1}(1)$ with $\|\dot{\gamma}\| \equiv 1$, then $o \in \mathbb{R}^n$ is non-singular of \tilde{F} .*

Proof. Since

$$\|\dot{c}(t)\| = \lim_{h \rightarrow 0} \frac{\|c(t+h) - c(t)\|}{|h|} = \lim_{h \rightarrow 0} \frac{\|\sigma(\gamma(t+h)) - \sigma(\gamma(t))\|}{\|\gamma(t+h) - \gamma(t)\|} \cdot \frac{\|\gamma(t+h) - \gamma(t)\|}{|h|},$$

$$(3.19) \quad \text{Lip}^b(\sigma)^{-1} \leq \|\dot{c}\| \leq \text{Lip}^b(\sigma)$$

holds on any geodesic segment $\gamma([0, \pi]) \subset S^{n-1}(1)$. By Lemma 3.10, (1.6), and (3.19),

$$\begin{aligned} \int_0^\pi e^{-t} \|\ddot{c}(t) + c(t)\| dt &= \int_0^\pi e^{-t} \sqrt{\|\ddot{c}(t)\|^2 - 2\|\dot{c}(t)\|^2 + 1} dt \\ &\leq \int_0^\pi e^{-t} \sqrt{\|\ddot{c}(t)\|^2 - 2\text{Lip}^b(\sigma)^{-2} + 1} dt \\ &\leq \int_0^\pi e^{-t} dt \sqrt{2 \left\{ \frac{\sqrt{2} - 1}{2(e^\pi - 1)} \right\}^2} \\ &= (1 - e^{-\pi}) \cdot \frac{2 - \sqrt{2}}{2(e^\pi - 1)} \\ &= e^{-\pi} \left(1 - \frac{1}{\sqrt{2}} \right). \end{aligned}$$

Hence, σ satisfies (3.1) for $\alpha = 1 - 1/\sqrt{2}$. By Theorem 3.1, o is non-singular of \tilde{F} . □

3.2 Preliminary to the proof in assuming (1.7)

Lemma 3.12 *Let $\sigma : S^{n-1}(1) \longrightarrow S^{n-1}(1)$ be a diffeomorphism, where we do not assume it to be (3.1), $\text{Lip}^b(\sigma)$ the bi-Lipschitz constant of it, and $\tilde{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a map defined by (3.14). Then, we have*

$$(3.20) \quad \text{Lip}^b(\sigma)^{-1} \|u - v\| \leq \|\tilde{F}(u) - \tilde{F}(v)\| \leq \text{Lip}^b(\sigma) \|u - v\|$$

for all $u, v \in \mathbb{R}^n$, i.e., \tilde{F} is bi-Lipschitz.

Proof. Take any $u, v \in \mathbb{R}^n$. If $u = 0$, or $v = 0$, then the inequality holds. Hence, we assume $u \neq 0$ and $v \neq 0$. Set $\tilde{u} := u/\|u\|$ and $\tilde{v} := v/\|v\|$. Note that if $\tilde{u} = \tilde{v}$, then (3.20) trivially holds. Hence, we assume $\tilde{u} \neq \tilde{v}$. Since $\|\tilde{F}(u)\| = \|u\|$ and $\langle \tilde{F}(u), \tilde{F}(v) \rangle = \|u\| \cdot \|v\| \langle \sigma(\tilde{u}), \sigma(\tilde{v}) \rangle$,

$$(3.21) \quad \|\tilde{F}(u) - \tilde{F}(v)\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\| \cdot \|v\| \langle \sigma(\tilde{u}), \sigma(\tilde{v}) \rangle$$

Since $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\| \cdot \|v\| \langle \tilde{u}, \tilde{v} \rangle$, we have, by (3.21),

$$(3.22) \quad \|\tilde{F}(u) - \tilde{F}(v)\|^2 - \|u - v\|^2 = 2\|u\| \cdot \|v\| (\langle \tilde{u}, \tilde{v} \rangle - \langle \sigma(\tilde{u}), \sigma(\tilde{v}) \rangle)$$

Similarly, we see

$$(3.23) \quad \|\sigma(\tilde{u}) - \sigma(\tilde{v})\|^2 - \|\tilde{u} - \tilde{v}\|^2 = 2(\langle \tilde{u}, \tilde{v} \rangle - \langle \sigma(\tilde{u}), \sigma(\tilde{v}) \rangle)$$

Set $\ell^2 := \|\sigma(\tilde{u}) - \sigma(\tilde{v})\|^2 / \|\tilde{u} - \tilde{v}\|^2$. Then, by (3.22) and (3.23),

$$(3.24) \quad \begin{aligned} \|\tilde{F}(u) - \tilde{F}(v)\|^2 - \|u - v\|^2 &= \|u\| \cdot \|v\| \{ \|\sigma(\tilde{u}) - \sigma(\tilde{v})\|^2 - \|\tilde{u} - \tilde{v}\|^2 \} \\ &= \|u\| \cdot \|v\| (\ell^2 - 1) \|\tilde{u} - \tilde{v}\|^2 \\ &= 2(\ell^2 - 1)(\|u\| \cdot \|v\| - \langle u, v \rangle). \end{aligned}$$

Since $\text{Lip}^b(\sigma) \geq 1$, $\|u\| \cdot \|v\| - \langle u, v \rangle \geq 0$ and $2(\|u\| \cdot \|v\| - \langle u, v \rangle) \leq \|u - v\|^2$, we see, by (3.24),

$$\begin{aligned} \|\tilde{F}(u) - \tilde{F}(v)\|^2 &= \|u - v\|^2 + 2(\ell^2 - 1)(\|u\| \cdot \|v\| - \langle u, v \rangle) \\ &\leq \|u - v\|^2 + 2(\text{Lip}^b(\sigma)^2 - 1)(\|u\| \cdot \|v\| - \langle u, v \rangle) \\ &= \text{Lip}^b(\sigma)^2 \|u - v\|^2 + (1 - \text{Lip}^b(\sigma)^2) \{ \|u - v\|^2 - 2(\|u\| \cdot \|v\| - \langle u, v \rangle) \} \\ &\leq \text{Lip}^b(\sigma)^2 \|u - v\|^2. \end{aligned}$$

As well as above, we have $\|\tilde{F}(u) - \tilde{F}(v)\|^2 \geq \text{Lip}^b(\sigma)^{-2} \|u - v\|^2$. □

3.3 The proofs

All notations in this subsection are those same defined in Sect. 1.3.

(Proof of the differentiable exotic sphere theorem I in the case where (1.6)) We assume here $d_{M_i}(p_i, q_i) := \pi$ for each $i = 1, 2$ by normalizing the metric, where d_{M_i} denotes the distance function on M_i . Note that the diffeomorphism $\sigma_{q_i}^{p_i} : \mathbb{S}_{p_i}^{n-1} \longrightarrow \mathbb{S}_{q_i}^{n-1}$ defined by (1.2) satisfies $\sigma_{q_i}^{p_i}(u_i) = -\dot{\tau}_{u_i}(\pi)$, i.e., $\ell_i = \pi$, since $d_{M_i}(p_i, q_i) = \pi$. Here, $\tau_{u_i}(t) := \exp_{p_i} t u_i$ for all $u_i \in \mathbb{S}_{p_i}^{n-1}$ and all $t \in [0, \pi]$.

We first construct a map \tilde{F} satisfying (3.14) under identifying $T_{q_i}M_i = \mathbb{R}^n$: For each $(t, u_1) \in [0, \pi] \times \mathbb{S}_{p_1}^{n-1}$, we define a bi-Lipschitz homeomorphism $F : M_1 \longrightarrow M_2$ by

$$(3.25) \quad F(\exp_{p_1} t u_1) := \exp_{p_2}(t I(u_1)),$$

where $I : T_{p_1}M_1 \longrightarrow T_{p_2}M_2$ denotes a linear isometry. Note that F is a diffeomorphism between $M_1 \setminus \{q_1\}$ and $M_2 \setminus \{q_2\}$. Define a map $\tilde{F} : \mathbb{B}_\pi(o_{q_1}) \longrightarrow \mathbb{B}_\pi(o_{q_2})$ by

$$(3.26) \quad \tilde{F} := \exp_{q_2}^{-1} \circ F \circ \exp_{q_1},$$

where for each $i = 1, 2$, $\mathbb{B}_\pi(o_{q_i}) := \{v \in T_{q_i}M_i \mid \|v\| < \pi\}$ and o_{q_i} denotes the origin of $T_{q_i}M_i$. Since

$$F(\exp_{q_1} t \sigma_{q_1}^{p_1}(u_1)) = F(\tau_{u_1}(\pi - t)) = \exp_{p_2}(\pi - t) I(u_1) = \tau_{u_2}(\pi - t) = \exp_{q_2} t \sigma_{q_2}^{p_2}(u_2),$$

where $u_2 := I(u_1)$, we have

$$(3.27) \quad \tilde{F}(t \sigma_{q_1}^{p_1}(u_1)) = \exp_{q_2}^{-1}(\exp_{q_2} t \sigma_{q_2}^{p_2}(u_2)) = t \sigma_{q_2}^{p_2}(u_2) = t \sigma_{q_2}^{p_2} \circ I(u_1)$$

By setting $v_1 := \sigma_{q_1}^{p_1}(u_1)$, it follows from (3.27) that

$$\tilde{F}(t v_1) = \tilde{F}(t \sigma_{q_1}^{p_1}(u_1)) = t \sigma_{q_2}^{p_2} \circ I(u_1) = t \sigma_{q_2}^{p_2} \circ I \circ \sigma_{p_1}^{q_1}(v_1) = t \sigma(v_1),$$

and hence \tilde{F} satisfies (3.14).

Since $d(\exp_{q_1}^{-1})_{q_1}$ and $d(\exp_{q_2}^{-1})_{q_2}$ are the identity maps, respectively, $\partial F(q_1) = \partial \tilde{F}(o)$, where $o := o_{q_1}$. By Lemma 3.7, $\partial F(q_1) = \text{Conv}(\{A_v \mid v \in \mathbb{S}_{q_1}^{n-1}\})$ holds, where A_v is as in the statement. Since $\text{Lip}^b(\sigma)$ and $\|\ddot{c}\|$ satisfy (1.6), by Corollary 3.11, q_1 is non-singular of F . Since F has no singular points on M_1 , it follows from Theorem 1.3 that for any sufficiently small $\eta > 0$, there exists a smooth approximation $f_\eta : M_1 \longrightarrow M_2$ of F , which is an immersion. Since $f_\eta(M_1)$ is open and closed in M_2 , $f_\eta(M_1) = M_2$. On the other hand, since M_1 is compact and M_2 is Hausdorff, f_η is proper. Since f_η is a local diffeomorphism, f_η is a covering map from M_1 onto M_2 . Since M_2 is simply connected, f_η is injective. Therefore, for any sufficiently small $\eta > 0$, f_η is a diffeomorphism from M_1 onto M_2 . \square

(Proof of the differentiable exotic sphere theorem I in the case where (1.7)) Let $F : M_1 \longrightarrow M_2$ and $\tilde{F} : \mathbb{B}_\pi(o_{q_1}) \longrightarrow \mathbb{B}_\pi(o_{q_2})$ be the maps defined by (3.25) and (3.26), respectively. Moreover, let M_2 be isometrically embedded into \mathbb{R}^m , where $m \geq n + 1$, by introducing the induced metric from the space. Then, $F : M_1 \longrightarrow \mathbb{R}^m$. By Lemma 3.12 and (1.7), we get $\text{Lip}^b(\tilde{F})^2 \leq 1 + \{(8/\pi)(n-1)\}^{-1/2}$. Hence, it follows from the proof of [21, Theorem 5.1] that the smooth approximation \tilde{F}_ε , which is the convolution of \tilde{F} and the

mollifier ρ_ε , is an immersion from some open ball $\mathbb{B}_\delta(o_{q_1}) \subset \mathbb{B}_\pi(o_{q_1})$ into $\mathbb{B}_\pi(o_{q_2})$. Here, $\tilde{F}_\varepsilon(y) := \int \tilde{F}(x)\rho_\varepsilon(x-y)dx$ for $\varepsilon < \delta$. Therefore, F admits a local smooth approximation $F_\varepsilon^{(q_1)}$ from an open ball $B_\delta(q_1) = \exp_{q_1} \mathbb{B}_\delta(o_{q_1})$ into \mathbb{R}^m , which is an immersion. Let $\varphi : M_1 \rightarrow \mathbb{R}$ be a smooth function satisfying $0 \leq \varphi \leq 1$ on M_1 , $\varphi \equiv 1$ on $\overline{B_r(q_1)}$, and $\text{supp } \varphi \subset B_R(q_1)$, where $0 < r < R < \delta$. Define a map $F_\varepsilon : M_1 \rightarrow \mathbb{R}^m$ by $F_\varepsilon := (1 - \varphi)F + \varphi F_\varepsilon^{(q_1)}$. Since

$$F_\varepsilon = \begin{cases} F_\varepsilon^{(q_1)} & \text{on } \overline{B_r(q_1)}, \\ F & \text{on } M_1 \setminus \text{supp } \varphi, \end{cases}$$

F_ε is a local diffeomorphism from $\overline{B_r(q_1)} \cup (M_1 \setminus \text{supp } \varphi)$ into \mathbb{R}^m . Since F is smooth on $\overline{B_R(q_1)} \setminus B_r(q_1)$, it is easy to see that $F_\varepsilon^{(q_1)}$ uniformly converges to F on the compact set as ε goes to zero in the C^1 -topology. Thus, F_ε uniformly converges to F on the set as ε goes to zero in the C^1 -topology. Now, choose a small neighborhood \mathcal{N}_{M_2} of $M_2 \subset \mathbb{R}^m$ such that any $x \in \mathcal{N}_{M_2}$ admits a unique nearest point $\pi_{M_2}(x) \in M_2$. Then, the map $\pi_{M_2} : \mathcal{N}_{M_2} \rightarrow M_2$ is well-defined and smooth. Since $d\pi_{M_2}$ is injective, a map $G_\varepsilon := \pi_{M_2} \circ F_\varepsilon : M_1 \rightarrow M_2$ is a local diffeomorphism on M_1 for any sufficiently small $\varepsilon > 0$. Since $G_\varepsilon(M_1)$ is open and closed in M_2 , $G_\varepsilon(M_1) = M_2$. Moreover, since M_1 is compact and M_2 is Hausdorff, G_ε is proper, and thus G_ε is a covering map from M_1 onto M_2 . Since M_2 is simply connected, G_ε is injective, and hence it is a diffeomorphism from M_1 onto M_2 for any sufficiently small $\varepsilon > 0$. \square

(Proof of the differentiable exotic sphere theorem II) Let $F : M_1 \rightarrow M_2$ and $\tilde{F} : \mathbb{B}_\pi(o_{q_1}) \rightarrow \mathbb{B}_\pi(o_{q_2})$ be the maps defined by (3.25) and (3.26), respectively. By the same argument in the proof of the case where (1.6), we have $\partial F(q_1) = \partial \tilde{F}(o) = \text{Conv}(\{A_v \mid v \in S^{n-1}_{q_1}\})$, where A_v is as in Lemma 3.7. Since σ satisfies (1.10), by Lemma 3.9, q_1 is non-singular of F , i.e., F has no singular points on M_1 . By Theorem 1.3 and the same argument in the proof of the case where (1.6), M_1 and M_2 are diffeomorphic. \square

3.4 The reason that we need the restriction of \ddot{c} in the condition (1.6)

Let $\sigma : S^{n-1}(1) \rightarrow S^{n-1}(1)$ be a diffeomorphism, where $S^{n-1}(1) := \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, and let $\text{Lip}^b(\sigma) \geq 1$ denote the bi-Lipschitz constant of σ defined by (1.4). First, we will prove that, for any $i, j \in \{1, 2, \dots, n\}$,

$$|\langle \sigma(e_i), \sigma(e_j) \rangle - \delta_{ij}| \leq \text{Lip}^b(\sigma)^2 - 1,$$

where $e_i := (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$. From this property, $\{\sigma(e_1), \sigma(e_2), \dots, \sigma(e_n)\}$ looks linearly independent if $\text{Lip}^b(\sigma)^2 - 1$ is sufficiently small. As Lemma 2.11 below shows, however, $\{\sigma(e_1), \sigma(e_2), \dots, \sigma(e_n)\}$ is not always linearly independent: Take any vectors $u, v \in S^{n-1}(1)$. By the parallelogram law, we have

$$\langle \sigma(u), \sigma(v) \rangle = \frac{1}{2} \{ \|\sigma(u)\|^2 + \|\sigma(v)\|^2 - \|\sigma(u) - \sigma(v)\|^2 \} = \frac{1}{2} \{ 2 - \|\sigma(u) - \sigma(v)\|^2 \}$$

and $\langle u, v \rangle = \{\|u\|^2 + \|v\|^2 - \|u - v\|^2\}/2 = \{2 - \|u - v\|^2\}/2$. Then, we have

$$\begin{aligned}\langle \sigma(u), \sigma(v) \rangle - \langle u, v \rangle &= \frac{1}{2} \{\|u - v\|^2 - \|\sigma(u) - \sigma(v)\|^2\} \\ &\leq \frac{1}{2} (1 - \text{Lip}^b(\sigma)^{-2}) \|u - v\|^2 \leq \frac{\text{Lip}^b(\sigma)^2 - 1}{2} \|u - v\|^2.\end{aligned}$$

As well as above, $\langle \sigma(u), \sigma(v) \rangle - \langle u, v \rangle \geq (1 - \text{Lip}^b(\sigma)^2) \|u - v\|^2/2$. Thus,

$$|\langle \sigma(u), \sigma(v) \rangle - \langle u, v \rangle| \leq \frac{\text{Lip}^b(\sigma)^2 - 1}{2} \|u - v\|^2$$

holds. In particular, $|\langle \sigma(e_i), \sigma(e_j) \rangle - \langle e_i, e_j \rangle| \leq \text{Lip}^b(\sigma)^2 - 1$.

Lemma 3.13 *For any $\varepsilon > 0$, there exist $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^n$ such that*

$$(3.28) \quad |\langle \mathbf{a}_i, \mathbf{a}_j \rangle - \delta_{ij}| < \varepsilon,$$

but $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is not linearly independent.

Proof. Take any $\varepsilon \in (0, 1)$, and fix it. Choose an integer $k \geq 0$ satisfying

$$(3.29) \quad \frac{4(1 - \varepsilon^2)}{\varepsilon^2} < k \leq \frac{4(1 - \varepsilon^2)}{\varepsilon^2} + 1.$$

Set $c_1 = c_2 = \dots = c_k = \varepsilon/2$. Since $\sum_{j=1}^k c_j^2 = k\varepsilon^2/4$, we have $1 - \sum_{j=1}^k c_j^2 = 1 - k\varepsilon^2/4$. By the right inequality of (3.29),

$$1 - \sum_{j=1}^k c_j^2 \geq 1 - \frac{\varepsilon^2}{4} \left\{ \frac{4(1 - \varepsilon^2)}{\varepsilon^2} + 1 \right\} = \frac{3}{4} \varepsilon^2 > 0.$$

By the left inequality of (3.29),

$$1 - \sum_{j=1}^k c_j^2 < 1 - \frac{\varepsilon^2}{4} \cdot \frac{4(1 - \varepsilon^2)}{\varepsilon^2} = \varepsilon^2.$$

Thus, we have $0 < c_{k+1} := \sqrt{1 - \sum_{j=1}^k c_j^2} < \varepsilon$. Defining $\mathbf{a}_1 := e_1, \mathbf{a}_2 := e_2, \dots, \mathbf{a}_k := e_k, \mathbf{a}_{k+1} = \sum_{j=1}^{k+1} c_j e_j \in \mathbb{R}^n$, we see $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = \delta_{ij}$ for all $i, j < k+1$, $\langle \mathbf{a}_{k+1}, \mathbf{a}_{k+1} \rangle = 1$, and $0 < \langle \mathbf{a}_{k+1}, \mathbf{a}_i \rangle < \varepsilon$ for all $i \neq k+1$. Hence, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k+1}\}$ is linearly dependent, but that satisfies (3.28). \square

Remark 3.14 By Lemma 3.13, it looks impossible to find the Lipschitz constant of \tilde{F} independent of the dimension n such that the differential of the smooth approximation of \tilde{F} at o is injective. In this sense, it is natural that the Lipschitz constant of the locally bi-Lipschitz map in [21, Theorem 5.1] depends on n , and it is too for (1.7).

4 Proof of the differentiable twisted sphere theorem

We need two lemmas in order to prove the differentiable twisted sphere theorem.

Lemma 4.1 *Let M be a complete Riemannian manifold, and let $f : M \rightarrow \mathbb{R}$ be a Lipschitz function on M . Then for any open neighborhood U of $f^{-1}(0)$, there exists a positive number ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, $f_\varepsilon^{-1}(0) \subset U$, where f_ε denotes a smooth approximation of f .*

Proof. For each $\varepsilon > 0$, let f_ε be a smooth approximation of the function f , i.e., $f_\varepsilon(q) := \sum_i \psi_i(q) f_\varepsilon^{(p_i)}(q)$, where $\{\psi_i\}_{i=1}^\ell$ denotes a partition of unity subordinate to a locally finite cover $\{B_{r_i}(p_i)\}$ of strongly convex balls $B_{r_i}(p_i) \subset M$, so that $\text{supp } \psi_i \subset B_{r_i}(p_i)$ and the local approximation $f_\varepsilon^{(p_i)}$ is defined by the equation (2.1). Since $\sum_{i=1}^\ell \psi_i(q) = 1$ on $f^{-1}(0)$, we have, by the triangle inequality,

$$(4.1) \quad |f_\varepsilon(q) - f(q)| = \left| \sum_{i=1}^\ell \psi_i(q) (f_\varepsilon^{(p_i)}(q) - f(q)) \right| \leq \sum_{i=1}^\ell \psi_i(q) |f_\varepsilon^{(p_i)}(q) - f(q)|.$$

Applying Lemma 2.2 to (4.1), we see that

$$|f_\varepsilon(q) - f(q)| \leq \varepsilon \cdot \text{Lip}(f) \sum_{i=1}^\ell \psi_i(q) \text{Lip}(\exp_{p_i} |_{\mathbb{B}_{r_i}(o_{p_i})})$$

for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \min\{\varepsilon_i \mid i = 1, 2, \dots, \ell\}$. Hence, for any sufficiently small $\varepsilon > 0$, $f_\varepsilon^{-1}(0)$ is a subset of U . \square

Lemma 4.2 *Let A, B be linear transformations on \mathbb{R}^n such that $A|_{\mathbb{R}^{n-1}} = B|_{\mathbb{R}^{n-1}} = \text{id}_{\mathbb{R}^{n-1}}$, and that $\langle A\vec{n}, \vec{n} \rangle > 0$, $\langle B\vec{n}, \vec{n} \rangle > 0$, where $\mathbb{R}^{n-1} := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$ and $\vec{n} := (0, \dots, 0, 1)$. Then, every element in $\text{Conv}(\{A, B\})$ is of maximal rank.*

Proof. Take any $\lambda \in [0, 1]$. Assume that there exists $\vec{v} + a\vec{n} \in \mathbb{R}^n$, where $a \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^{n-1}$, such that $(\lambda A + (1-\lambda)B)(\vec{v} + a\vec{n}) = o$. Since $o = (\lambda A + (1-\lambda)B)(\vec{v} + a\vec{n}) = \vec{v} + a(\lambda A + (1-\lambda)B)\vec{n}$, we have

$$0 = \langle \vec{v} + a(\lambda A + (1-\lambda)B)\vec{n}, \vec{n} \rangle = a(\lambda \langle A\vec{n}, \vec{n} \rangle + (1-\lambda) \langle B\vec{n}, \vec{n} \rangle),$$

and hence $a = 0$. Since $(\lambda A + (1-\lambda)B)(\vec{v} + a\vec{n}) = o$, we have $\vec{v} = o$. Thus, $\vec{v} + a\vec{n} = o$. This implies that $\lambda A + (1-\lambda)B$ is non-singular for any $\lambda \in [0, 1]$. \square

In what follows, all notations are those same defined in Sect. 1.3.

(*Proof of the differentiable twisted sphere theorem*) We first prove (T-1), i.e., M is a twisted sphere: Let d_p, d_q be the distance functions to p, q , respectively, i.e., $d_p(x) := d(p, x)$ for all $x \in M$. Consider a Lipschitz function $f := d_p - d_q$ on M . Remark that $f^{-1}(0) = E_{p,q}$ is compact. Let f_ε denote a smooth approximation of f , i.e., $f_\varepsilon = (d_p)_\varepsilon - (d_q)_\varepsilon$. Applying Lemma 2.14 to the compact set $K = E_{p,q}$, we obtain that $\nabla f_\varepsilon \neq 0$ on $E_{p,q}$ for all sufficiently small $\varepsilon > 0$. Choose any $R > 0$ so as to be $p, q \notin B_R(f^{-1}(0))$. Here

$B_R(f^{-1}(0)) := \{x \in M \mid \tilde{d}(f^{-1}(0), x) < R\}$, where $\tilde{d}(f^{-1}(0), x) = \min_{y \in f^{-1}(0)} d(y, x)$. By Lemma 4.1, $f_\varepsilon^{-1}(0) \subset B_R(f^{-1}(0))$ for any sufficiently small $\varepsilon > 0$, and hence $f_\varepsilon^{-1}(0)$ is a regular compact hypersurface. Choose such a sufficiently small ε , and fix it in the following. Now, let $\overline{D_\varepsilon(p)} := \{x \in M \mid f_\varepsilon(x) \leq 0\}$ and $\overline{D_\varepsilon(q)} := \{x \in M \mid f_\varepsilon(x) \geq 0\}$. Since d_p is smooth on a punctured convex ball $\mathcal{P}(p)$ at p , $\|\nabla(d_p)_\varepsilon - \nabla d_p\|$ is sufficiently small on $\mathcal{P}(p)$. Therefore, we can assume that there exists a non-zero smooth vector field X_+ on $\overline{D_\varepsilon(p)} \setminus \{p\}$ such that

$$X_+ = \begin{cases} \nabla d_p & \text{on } \mathcal{P}(p), \\ \nabla(d_p)_\varepsilon & \text{on a neighborhood of } f_\varepsilon^{-1}(0). \end{cases}$$

For each $v \in \mathbb{S}_p^{n-1}$, let $\tau_+(t, v)$ be the integral curve of X_+ with initial conditions $\tau_+(0, v) = p$ and $\frac{\partial \tau_+}{\partial t}(0, v) = v$. Since $X_+ = \nabla d_p$ on $\mathcal{P}(p)$, $\tau_+(t, v) = \exp_p tv$ holds for all sufficiently small $t \geq 0$. It follows from Lemma 2.14 that $\angle(\nabla(d_p)_\varepsilon, \nabla(d_q)_\varepsilon) > \pi/2$ on $f_\varepsilon^{-1}(0)$. Thus, there exists a smooth solution $t = t_+(v) > 0$ of $f_\varepsilon(\tau_+(t, v)) = 0$, since X_+ is not tangent to $f_\varepsilon^{-1}(0)$ at each point of the hypersurface. Then, we have a diffeomorphism G_+ from $\overline{D_\varepsilon(p)}$ onto $S_+^n(1) := \{(x_1, x_2, \dots, x_{n+1}) \in S^n(1) \mid x_{n+1} \geq 0\}$ defined by

$$G_+(\tau_+(t, v)) := \exp_N \frac{t\pi}{2t_+(v)} I(v)$$

for all $(t, v) \in [0, \infty) \times \mathbb{S}_p^{n-1}$ with $t \leq t_+(v)$, where I denotes a linear isometry from $T_p M$ onto $T_N S^n(1)$. In the same way, we have a diffeomorphism G_- from $\overline{D_\varepsilon(q)}$ onto $S_-^n(1) := \{(x_1, x_2, \dots, x_{n+1}) \in S^n(1) \mid x_{n+1} \leq 0\}$. Thus, we can define the induced metrics $g_\pm := G_\pm^* g_0$ on $\overline{D_\varepsilon(p)}, \overline{D_\varepsilon(q)}$ from $S^n(1)$, where g_0 denotes the metric of $S^n(1)$, so that $\overline{D_\varepsilon(p)}, \overline{D_\varepsilon(q)}$ are isometric to $S_+^n(1), S_-^n(1)$, respectively. Let \exp^{g_+}, \exp^{g_-} be exponential maps on tangent spaces at p, q of $\overline{D_\varepsilon(p)}, \overline{D_\varepsilon(q)}$ with respect to g_\pm , respectively. For each $(t, v) \in [0, \pi] \times \mathbb{S}_p^{n-1}$, we define a point $e(t, v)$ on M with respect to g_\pm by

$$e(t, v) = \begin{cases} \exp^{g_+}(tv) & \text{on } [0, \pi/2], \\ \exp^{g_-}((\pi - t)\sigma_q^p(v)) & \text{on } [\pi/2, \pi], \end{cases}$$

where $\sigma_q^p : \mathbb{S}_p^{n-1} \rightarrow \mathbb{S}_q^{n-1}$ is the diffeomorphism satisfying $\exp^+(\pi v/2) = \exp^-(\pi \sigma_q^p(v)/2)$. Thus, we have a boundary diffeomorphism $h_{\sigma_q^p} : \partial S_+^n(1) \rightarrow \partial S_-^n(1)$ induced from σ_q^p . Hence, $M = S_+^n(1) \cup_{h_{\sigma_q^p}} S_-^n(1)$ is twisted.

We next prove (T-2), i.e., we construct a bi-Lipschitz homeomorphism from M to $S^n(1)$ that admits a diffeomorphism between $M \setminus \{q\}$ and $S^n(1) \setminus \{S\}$: For every $(t, v) \in [0, \pi] \times \mathbb{S}_p^{n-1}$, define a map $F : M \rightarrow S^n(1)$ by

$$F(e(t, v)) := \exp_N tI(v).$$

It is not difficult to see that F is bi-Lipschitz. Since F is a local diffeomorphism on $M \setminus (f_\varepsilon^{-1}(0) \cup \{q\})$, F has no singular points on the open set. Let F^+ and F^- be the smooth extensions of $F|_{\overline{D_\varepsilon(p)}}$ and $F|_{\overline{D_\varepsilon(q)} \setminus \{q\}}$, respectively. Since $\partial F(x) = \text{Conv}(\{dF_x^+, dF_x^-\})$ for each $x \in f_\varepsilon^{-1}(0)$, it follows from Lemma 4.2 that any element in $\partial F(x)$ is of maximal

rank. Hence, F has no singular points on $M \setminus \{q\}$. Since $B_R(f^{-1}(0))$ is open in M_1 and $f_\varepsilon^{-1}(0) \subset B_R(f^{-1}(0))$, there exists $r \in (0, R)$ such that $B_r(f_\varepsilon^{-1}(0)) \subset B_R(f^{-1}(0))$. Note that $p, q \notin \overline{B_r(f_\varepsilon^{-1}(0))}$. Let φ be a smooth function on M satisfying $0 \leq \varphi \leq 1$ on M , $\varphi \equiv 1$ on $\overline{B_r(f_\varepsilon^{-1}(0))}$, and $\text{supp } \varphi \subset B_R(f^{-1}(0))$. Define a map $G_\varepsilon : M \rightarrow \mathbb{R}^{n+1}$ by $G_\varepsilon := (1 - \varphi)F + \varphi F_\varepsilon$, where $F_\varepsilon : M \rightarrow \mathbb{R}^{n+1}$ denotes the smooth approximation of F defined by (2.16). It is clear that

$$G_\varepsilon = \begin{cases} F_\varepsilon & \text{on } \overline{B_r(f_\varepsilon^{-1}(0))}, \\ F & \text{on } M \setminus \text{supp } \varphi. \end{cases}$$

Since F has no singular points on $\overline{B_r(f_\varepsilon^{-1}(0))}$, by the same argument in the proof of Lemma 2.25, F_ε is a smooth immersion on the compact set. Since $\varepsilon > 0$ is sufficiently small, by (4.1), we can assume that $|f_\varepsilon - f| \leq r$ on M . Then, since F is smooth on $\overline{B_R(f^{-1}(0))} \setminus B_r(f_\varepsilon^{-1}(0))$, F_ε uniformly converges to F on the compact set as $\varepsilon \downarrow 0$ in the C^1 -topology. Thus, we see that G_ε uniformly converges to F on the compact set as $\varepsilon \downarrow 0$ in the C^1 -topology. Hence, $G_\varepsilon : M \rightarrow \mathbb{R}^{n+1}$ is a bi-Lipschitz homeomorphism which is a local diffeomorphism on $M \setminus \{q\}$. Define a map $\psi_\varepsilon : M \rightarrow S^n(1)$ by $\psi_\varepsilon := \pi_{S^n(1)} \circ G_\varepsilon$, where $\pi_{S^n(1)} : \mathbb{R}^{n+1} \setminus \{o\} \rightarrow S^n(1)$ denotes the distance projection. By the similar argument in the proof of the differentiable exotic sphere theorem, we see that ψ_ε is a covering map from M onto $S^n(1)$. Since $S^n(1)$ is simply connected, ψ_ε is injective. Therefore, ψ_ε is a bi-Lipschitz homeomorphism from M onto $S^n(1)$ which is a diffeomorphism except for q .

Finally, we prove (T-3): Define a map $\tilde{F} : \mathbb{B}_\pi(o_q) \rightarrow T_S S^n(1)$ by $\tilde{F} := \exp_S^{-1} \circ F \circ \exp^{g^-}$. By the similar argument in the proof of the differentiable exotic sphere theorem, we then see that $\tilde{F}(tv) = t\sigma_S^N \circ I \circ (\sigma_q^p)^{-1}(v)$ for all $(t, v) \in [0, \pi] \times \mathbb{S}_q^{n-1}$. Here, σ_S^N denotes a diffeomorphism defined by (1.2). In particular, \tilde{F} satisfies (3.14). Hence, by Lemma 3.7, $\partial F(q) = \text{Conv}(\{A_v \mid v \in \mathbb{S}_q^{n-1}\})$, where A_v is as in the lemma. For a geodesic segment $\gamma : [0, \pi] \rightarrow \mathbb{S}_q^{n-1}$, let $c : [0, \pi] \rightarrow \mathbb{S}_S^{n-1}$ be a curve defined by $c := \sigma_S^N \circ I \circ (\sigma_q^p)^{-1} \circ \gamma$. Now, we assume $\text{Lip}^b((\sigma_q^p)^{-1})$ and $\|\dot{c}\|$ satisfy (1.6) for all geodesic segments $\gamma([0, \pi]) \subset \mathbb{S}_q^{n-1}$ with $\|\dot{\gamma}\| \equiv 1$. Since

$$\text{Lip}^b(\sigma_S^N \circ I \circ (\sigma_q^p)^{-1}) = \text{Lip}^b((\sigma_q^p)^{-1}),$$

$\text{Lip}^b(\sigma_S^N \circ I \circ (\sigma_q^p)^{-1})$ and $\|\dot{c}\|$ satisfy (1.6). By Corollary 3.11, q is non-singular of F , and hence F has no singular points on M . Therefore, it follows from Theorem 1.3 that M and $S^n(1)$ are diffeomorphic. For the cases where (1.7) and (1.10), we can prove that M and $S^n(1)$ are diffeomorphic by the same arguments in Sect. 3.3, respectively. \square

Remark 4.3 By applying [37, Proposition C] to the two discs $\overline{D_\varepsilon(p)}$ and $\overline{D_\varepsilon(q)}$ in the proof above successively, we get a new Riemannian metric on M , points $p_1 \in D_\varepsilon(p)$ and $q_1 \in D_\varepsilon(q)$ such that all geodesics (with this new metric) emanating from p_1 (respectively q_1) pass through the boundary of $\overline{D_\varepsilon(p)}$ (respectively $\overline{D_\varepsilon(q)}$) perpendicularly. Therefore, any geodesic emanating from p_1 passes through q_1 , i.e., the cut locus of p_1 consists of q_1 .

Remark 4.4 The proofs of Corollaries 1.15, 1.17, and 1.19 are found in Sect. 1.3.

Acknowledgements. The authors express their sincere thanks to Professors K. Grove, P. Petersen, and F. Wilhelm, who pointed out a mistake in the first version, back then entitled “A sufficient condition for a pair of bi-Lipschitz homeomorphic manifolds to be diffeomorphic and sphere theorems”, of this article. Thanks to the indication, they could arrive at the differentiable exotic sphere theorem. The first named author would like to thank to Professor T. Shioya, who suggested the shorter proof of Lemma 2.26 than that in the first version. He is deeply grateful to Professor F.H. Clarke for his encouragement, to Professors M. Gromov, S. Ohta, and T. Yamaguchi for their comments on the first one, and finally to Professors Y. Agaoka, M. Ishida, and K. Yasui for their having given knowledge about exotic structures, which has had a big influence on the structure of Sect. 1.

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